INTENSE AUTOMORPHISMS OF GROUPS

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FRAUEN IN DER MATHEMATIK
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Joint work with

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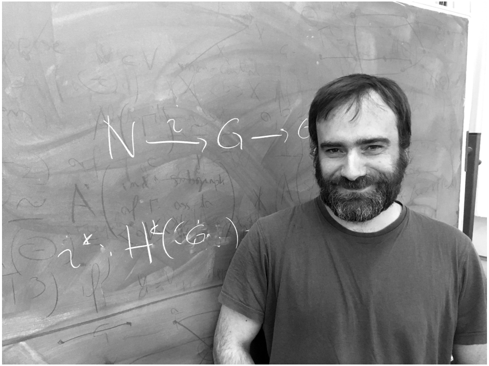

Jon González Sánchez (EHU)

Groups and symmetries

Symmetries of objects

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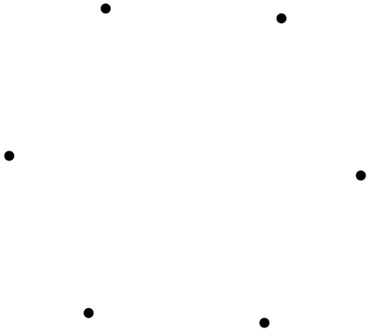
- $\mathrm{Gal}(\ell/k)$
- $\mathrm{Aut}(G)$

Permutation of points



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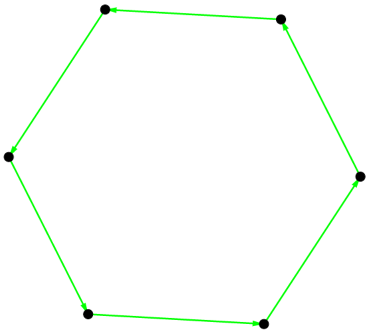

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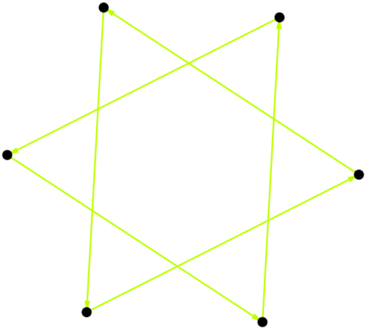

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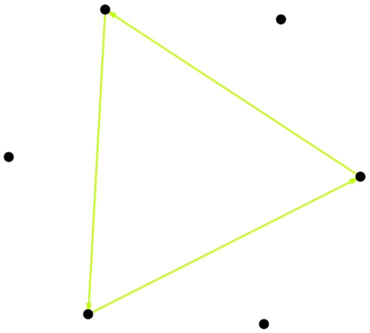

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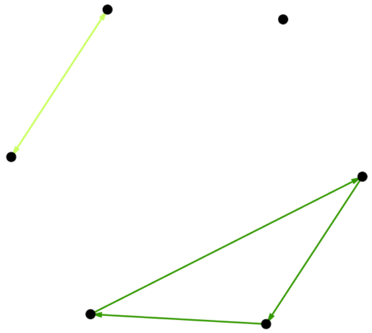

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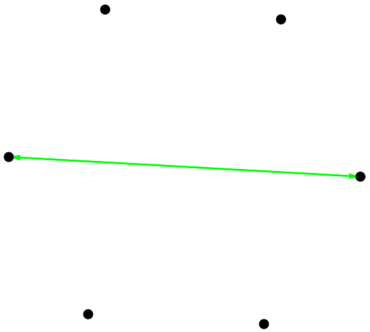

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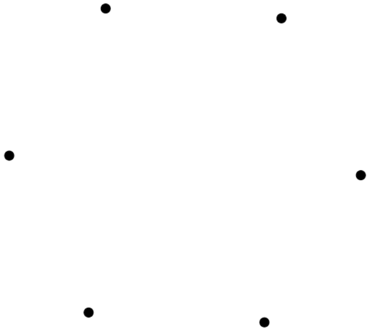

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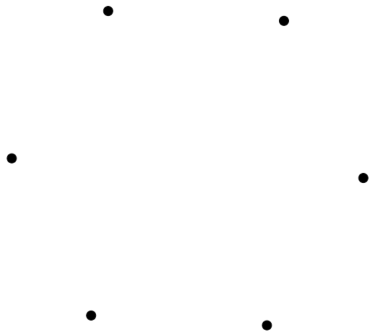

The group of symmetries is $\mathrm{Sym}(X) = S_6$

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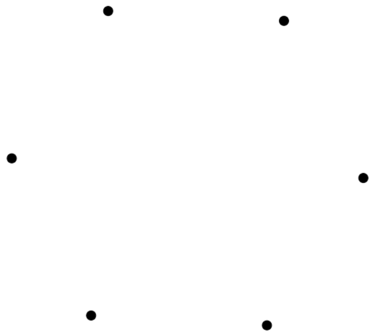
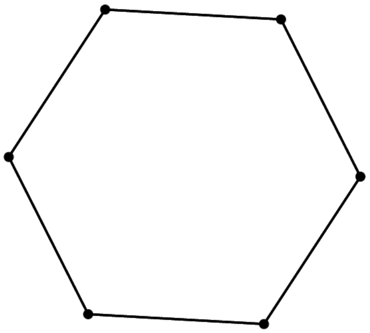


The group of symmetries is $\mathrm{Sym}(X) = S_6$ which has 720 elements.

Symmetries of a regular polygon



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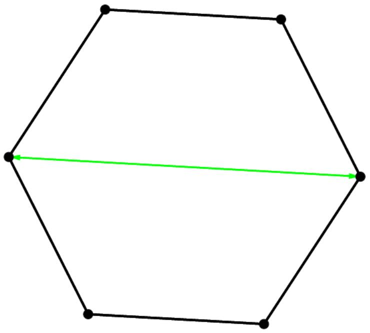

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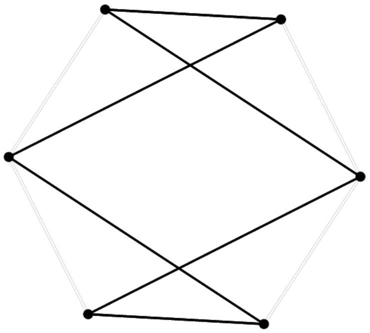

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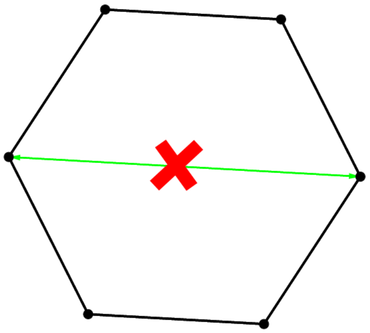

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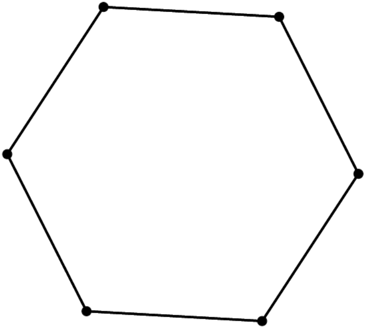

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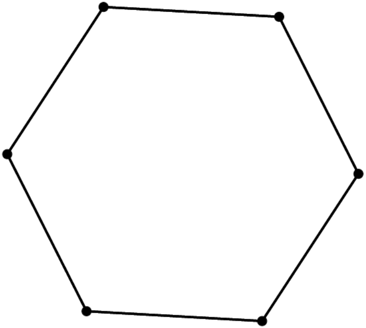

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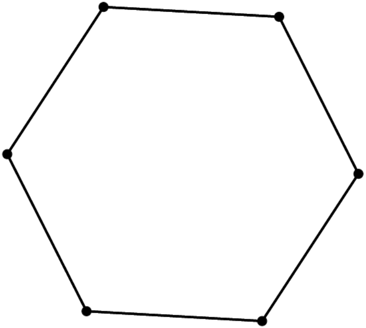
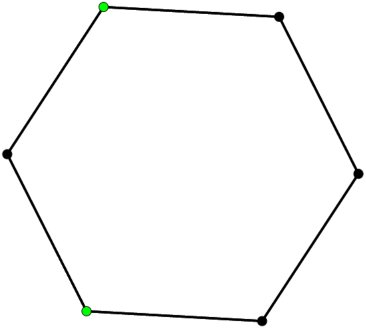


The group of symmetries is $\mathrm{Sym}(X) = \mathrm{D}_{12}$ which has 12 elements.

Symmetries of a coloured polygon



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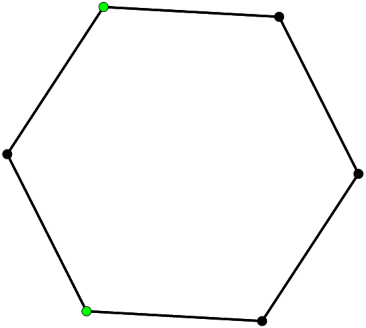

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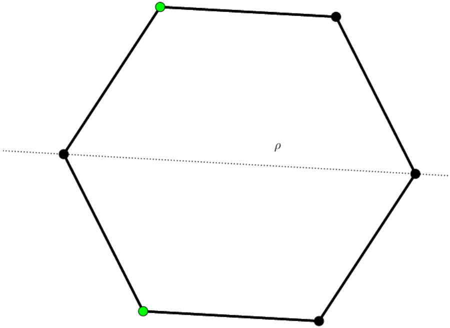

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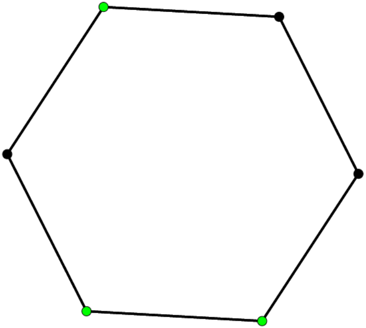

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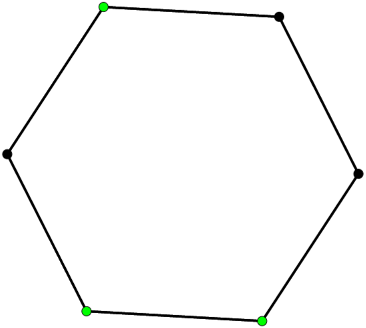

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If V has finite dimension n over k, then $GL_k(V) \cong \mathrm{GL}_n(k)$.

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If the extension ℓ/k is Galois, then $\mathrm{Gal}(\ell/k) = \mathrm{Aut}_k(\ell)$.

Symmetries of groups

Let G be a group. Then $\mathrm{Sym}(G) = \mathrm{Aut}(G)$ consists of all maps $\alpha : G \to G$ such that:

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Example:

- $\mathrm{Aut}(\{1\}) = \{\mathrm{id}\}$;
- $\mathrm{Aut}(\mathbb{Z}) = \{\pm\,\mathrm{id}\}$;
- $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^*$.

From the symmetries to the group

Example 1

Let G be a group. Then

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Let G be a finitely generated group. Then

$\mathrm{Aut}(G)$ is cyclic $\Rightarrow$ G is one between

- $\mathbb{Z}/2\mathbb{Z}$;
- $\mathbb{Z}/4\mathbb{Z}$;
- $\mathbb{Z}/p^k\mathbb{Z}$, for p odd and $k \geq 0$;
- $\mathbb{Z}$.

Example 3

Let G be a group and assume that

- G has cardinality $729 = 3^6$.
- G is 2-generated.
- $\mathrm{Aut}(G)$ has cardinality 104976.

Then G is unique up to isomorphism.

Non-Example 3

Let G be a group and assume that

- G has cardinality $729 = 3^6$.
- G is 2-generated.
- ~~$\mathrm{Aut}(G)$ has cardinality 104976~~.

Then there are 100 possible isomorphism classes for G.

General idea

Conditions on the structure of $\mathrm{Aut}(G)$

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$$\downarrow$$

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PB: Does G even exist?

Intense automorphisms of groups

Intense automorphisms

Let G be a finite group. An automorphism α of G is **intense** if for all $H \leq G$ there exists $g \in G$ such that $\alpha(H) = gHg^{-1}$. Write $\alpha \in \mathrm{Int}(G)$.

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Motivation: Intense automorphisms appear naturally as solutions to a certain cohomological problem. They (surprisingly!) give rise to a very rich theory.

Example:

- Every automorphism of a cyclic group is intense.
- Inner automorphisms are intense.

Intensity

Let p be a prime number and let G be a finite p-group.

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The **intensity** of G is $\mathrm{int}(G) = \#C$.

The problem

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YES!

Intense triples

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There are no intense triples with $p = 2$.

Equivalent triples

Example:

Let p be an odd prime and let $n \in \mathbb{Z}_{>0}$.
For all $\alpha \in \mathbb{F}_p^* \setminus \{1\}$, the triple $(p, \mathbb{F}_p^n, \alpha)$ is intense.

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Two intense triples (p, G, α) and (q, G', β) are **equivalent** if there exists an isomorphism $\sigma : G \to G'$ such that $\beta = \sigma\alpha\sigma^{-1}$. It follows that $p = q$.

Let $\mathcal{T} = \{[p, G, \alpha] \mid p, G, \alpha \ldots\}$ denote the set of equivalence classes of intense triples.

Abelian groups

Let p be a prime number and let $\mathbb{Z}_p$ denote the ring of p-adic integers. Define $\omega(\mathbb{F}_p^*) = \{\alpha \in \mathbb{Z}_p^* \mid \alpha^{p-1} = 1\}$.

Note that $\omega(\mathbb{F}_p^*) \cong \mathbb{F}_p^*$ and that every abelian p-group has a natural structure of $\mathbb{Z}_p$-module.

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Proposition

Assume that:

- *p is odd.*
- *$G \neq 1$ is a finite abelian p-group.*
- *$\alpha \in \omega(\mathbb{F}_p^*) \setminus \{1\}$. (Example: $\alpha = -1$)*

Then $[p, G, \alpha] \in \mathcal{T}$ *and* $\mathrm{int}(G) = p - 1$.

The lower central series

Let G be a finite group.

- If $x, y \in G$, then $[x, y] = xyx^{-1}y^{-1}$.
- If $H, K \leq G$, then $[H, K] = \langle [x, y] \mid x \in H, y \in K \rangle$.

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If G is a p-group, there exists k such that $G_k = 1$ and the **(nilpotency) class** of G is

$$c = \#\{i \mid G_i \neq G_{i+1}\} = -1 + \min\{k \mid G_k = 1\}.$$

Strategy

For all $c \in \mathbb{Z}_{\geq 0}$, let $\mathcal{T}[c] = \{[p, G, \alpha] \in \mathcal{T} \mid G \text{ has class } c\}$.

Then:

- $\mathcal{T} = \bigsqcup_c \mathcal{T}[c]$.
- $\mathcal{T}[0] = \emptyset$.
- $\mathcal{T}[1] = \{[p, G, \alpha] \text{ as in the Proposition}\}$.
- $\mathcal{T}[c]$ for $c = 2$?
- $\mathcal{T}[c]$ for $c \geq 3$?

Class 2

Let p be an odd prime and let $n \in \mathbb{Z}_{>0}$. Define $(\mathrm{ES}(p,n), *)$ as

- $\mathrm{ES}(p,n) = \mathbb{F}_p \times \mathbb{F}_p^n \times \mathbb{F}_p^n$.
- $(z_1, y_1, x_1) * (z_2, y_2, x_2) = (z_1 + z_2 + x_1 \cdot y_2, y_1 + y_2, x_1 + x_2)$.

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<u>*Exercise:*</u>

- $(\mathrm{ES}(p,n),*)$ has order p^{2n+1} and class 2.
- Let $\lambda \in \mathbb{F}_p^*$. Then $\alpha_\lambda : (z,y,x) \mapsto (\lambda^2 z, \lambda y, \lambda x)$ is an intense automorphism of $(\mathrm{ES}(p,n),*)$.

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Proposition

$\mathcal{T}[2] = \{[p, (\mathrm{ES}(p,n),*), \alpha_\lambda] \mid p \textit{ is odd}, n \in \mathbb{Z}_{>0}, \lambda \in \mathbb{F}_p^* \setminus \{1\}\}$.

Class at least 3

Given a p-group G, let $(G_i)_{i\geq 1}$ be its lower central series. Let $(f_i)_{i\geq 1}$ be the sequence, with values in $\mathbb{Z}_{\geq 0}$, such that the order of G_i/G_{i+1} is equal to p^{f_i}.

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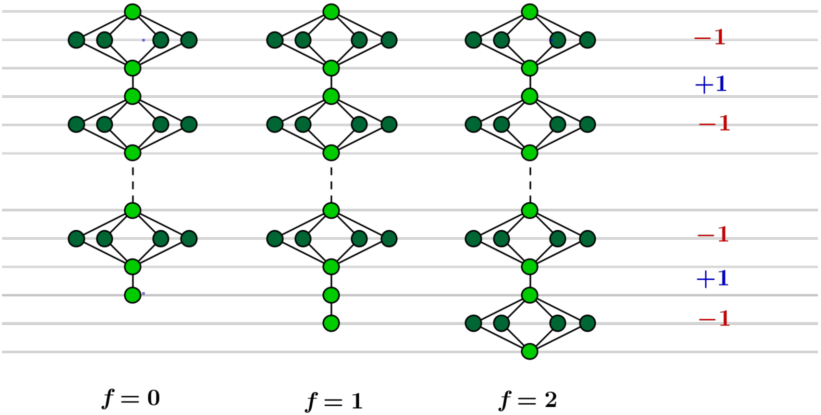
Proposition

Let $c \geq 3$ and assume $[p, G, \alpha] \in \mathcal{T}[c]$. The following hold.

- *The order of α is equal to 2 and* $\mathrm{int}(G) = 2$.
- *For all i, the quotient G_i/G_{i+1} is a vector space over $\mathbb{F}_p$ and α induces multiplication by $(-1)^i$ on it.*
- $(f_i)_{i\geq 1} = (2, 1, 2, 1, \dots, 2, 1, f, 0, 0, 0, \dots)$ *with* $f \in \{0, 1, 2\}$.

Normal subgroups structure



An intense graph

Fix p and define $\mathcal{T}_p = \{[p, G, \alpha] \mid G, \alpha, \dots\}$.

There is a well-defined sequence of sets

$$\dots \longrightarrow \mathcal{T}_p[c+1] \xrightarrow{\pi_{c+1}} \mathcal{T}_p[c] \xrightarrow{\pi_c} \mathcal{T}_p[c-1] \longrightarrow \dots$$

where, for all c, the map π_c is defined by

$$\pi_c : [p, G, \alpha] \mapsto [p, G/G_c, \overline{\alpha}].$$

An intense graph

Fix p and define $\mathcal{T}_p = \{[p, G, \alpha] \mid G, \alpha, \ldots\}$.

There is a well-defined sequence of sets

$$\ldots \longrightarrow \mathcal{T}_p[c+1] \xrightarrow{\pi_{c+1}} \mathcal{T}_p[c] \xrightarrow{\pi_c} \mathcal{T}_p[c-1] \longrightarrow \ldots$$

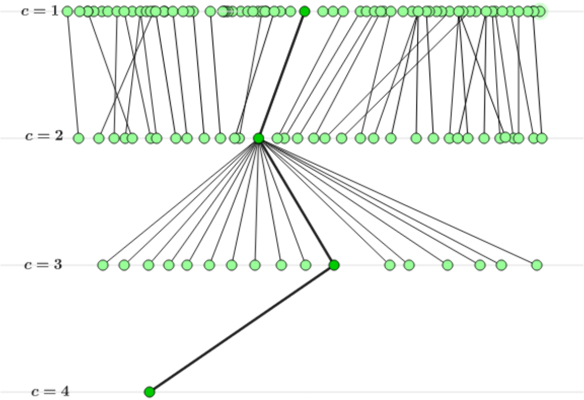
where, for all c, the map π_c is defined by

$$\pi_c : [p, G, \alpha] \mapsto [p, G/G_c, \overline{\alpha}].$$

We define a graph $\mathcal{G}_p = (E_p, V_p)$, where

- $V_p = \mathcal{T}_p$.
- $(v, w) \in E_p$ if there exists c such that $\pi_c(v) = w$.

The graph for $p = 3$



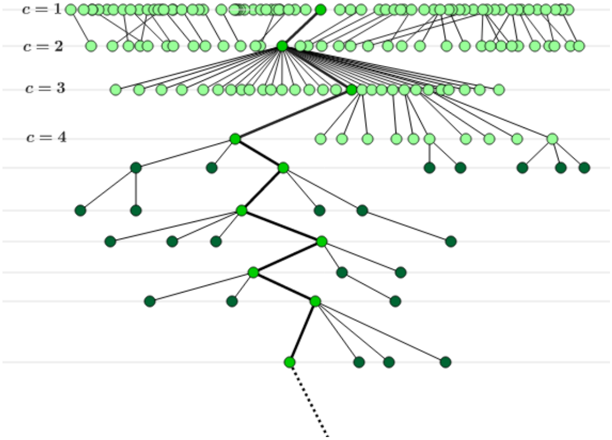
Example 3

Let G be a group. Assume that

- G has cardinality $729 = 3^6$.
- G is 2-generated.
- $\mathrm{Aut}(G)$ has cardinality 104976.

Then G is unique up to isomorphism.

The graph for $p > 3$



The infinite case

Theorem

Let p be an odd prime and let $c \in \mathbb{Z}_{>0}$. Then the following hold.

- *If $c \geq 3$, then $\mathcal{T}_p[c]$ is finite.*
- $\mathcal{T}_p[c] = \emptyset \iff p = 3$ *and* $c \geq 5$.
- *If $p > 3$, then $\# \varprojlim_{c} \mathcal{T}_p[c] = 1$.*

The infinite case

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If $\varprojlim_{c} \mathcal{T}_p[c] = \{[p, G^{(c)}, \alpha^{(c)}]\}_{c>0}$, we want to determine the pro-p-group $G_{lim} = \varprojlim_{c} G^{(c)}$ and the automorphism α_{lim} of G_{lim} that is induced by the automorphisms $\alpha^{(c)}$.

A profinite example

Let $p > 3$ be a prime and let $t \in \mathbb{Z}_p$ satisfy $(\frac{t}{p}) = -1$. Set $\mathrm{A}_p = \mathbb{Z}_p + \mathbb{Z}_p\mathrm{i} + \mathbb{Z}_p\mathrm{j} + \mathbb{Z}_p\mathrm{ij}$ with defining relations $\mathrm{i}^2 = t$, $\mathrm{j}^2 = p$, and $\mathrm{ji} = -\mathrm{ij}$. Then A_p is a non-commutative local ring such that $\mathrm{A}_p/\mathrm{j}\mathrm{A}_p \cong \mathbb{F}_{p^2}$. The involution $\overline{\cdot} : \mathrm{A}_p \to \mathrm{A}_p$ is defined by

$$a = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{ij} \;\mapsto\; \bar{a} = s - t\mathrm{i} - u\mathrm{j} - v\mathrm{ij}.$$

Let $G = \{a \in \mathrm{A}_p^* \mid a\bar{a} = 1 \text{ and } a \equiv 1 \bmod \mathrm{j}\mathrm{A}_p\}$ and, for all $a \in G$, define $\alpha(a) = \mathrm{i}a\mathrm{i}^{-1}$.

Theorem

G is a pro-p-group and α is topologically intense, i.e. for any closed subgroup H of G there exists $g \in G$ such that $\alpha(H) = gHg^{-1}$. Moreover, $(G, \alpha) \cong (G_{lim}, \alpha_{lim})$.

THANK YOU