Configurations of lines on del Pezzo surfaces

Rosa Winter
Universiteit Leiden

Konstanz Women in Mathematics Lecture Series

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Today: talk about a project that started as my master thesis.

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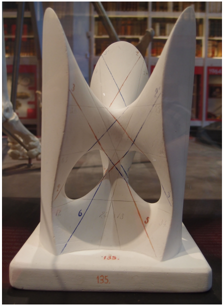
Example

$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3$ (Clebsch surface)



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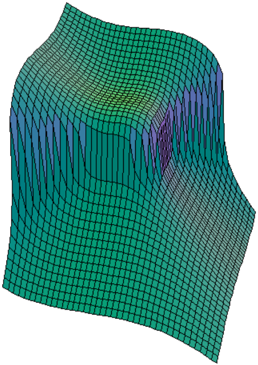
$x^3 + y^3 + z^3 = 1$ (Fermat cubic)



Cubic surfaces

Theorem (Cayley-Salmon, 1849)

- Such a surface contains exactly 27 lines.
- Any point on the surface is contained in at most three of those lines.

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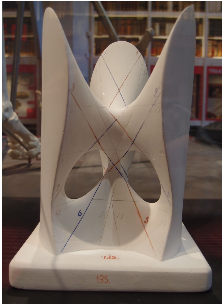
Clebsch surface



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A point on a smooth cubic surface in $\mathbb{P}^3$ that is contained in three lines is called an *Eckardt point*.

Lemma (Hirschfeld, 1967)
There are at most 45 Eckardt points on a cubic surface.

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Lemma (Hirschfeld, 1967)
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Example
The Clebsch surface has 10 Eckardt points; the Fermat cubic has 18 Eckardt points.

More general: del Pezzo surfaces

A smooth cubic surface is a surface given by an equation of degree 3 in 3-dimensional space. This is an example of a del Pezzo surface.

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Definition

A del Pezzo surface X is a 'nice' surface over a field k that has an embedding in some $\mathbb{P}_k^n$, such that $-aK_X$ is linearly equivalent to a hyperplane section for some a. The *degree* is the self intersection $(-K_X)^2$ of the anticanonical divisor.

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Question:
What do we know about lines on del Pezzo surfaces of other degrees? Generalizations of Eckardt points?

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Let P be a point in the plane. The construction blowing up replaces P by a line E, called the exceptional curve above P; each point on this line E is identified with a direction through P.

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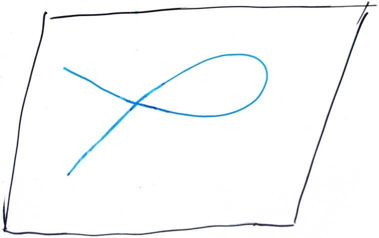
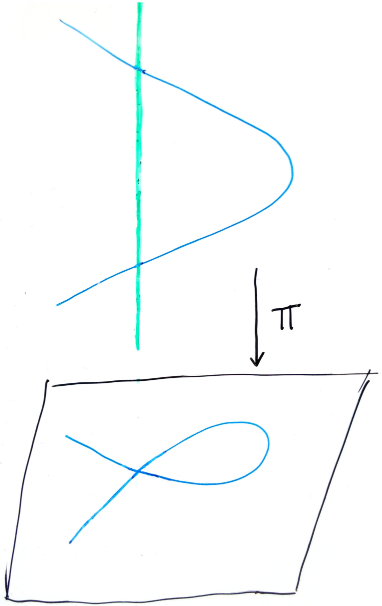
We often do this to *resolve a singularity*.

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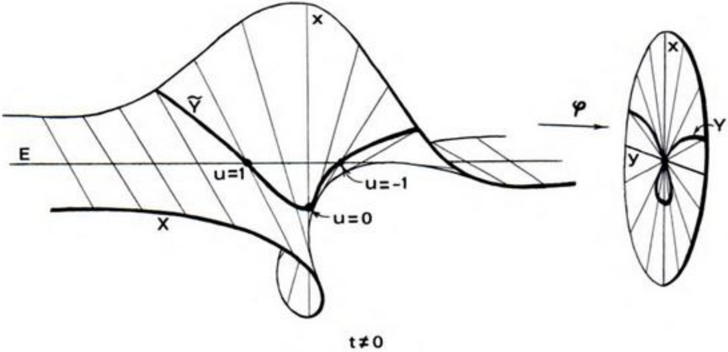

From: Robin Hartshorne, Algebraic Geometry.

Some facts about blow-ups of points

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- Two lines that intersect in the plane in P do not intersect in X! They both intersect the exceptional curve E, but in different points.
- Outside P, everything stays the same.

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<u>Theorem</u>
Let X be a del Pezzo surface of degree d over an algebraically closed field. Then X is isomorphic to either the product of two lines (only for degree 8), or $\mathbb{P}^2$ blown up in $9 - d$ points in general position.

where *general position* means

- no three points on a line;
- no six points on a conic;
- no eight points on a cubic that is singular at one of them.

The Picard group of a del Pezzo surface

Let k be an algebraically closed field, and let X be the blow up of $\mathbb{P}^2_k$ in points $P_1, \ldots, P_r$ $(1 \leq r < 9)$. Let E_i be the exceptional curve above P_i.

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Facts

- We have $E_i^2 = -1$ for all i.
- For $d = 9 - r \geq 3$, the lines on the embedding of X in $\mathbb{P}^d$ correspond to the classes C in Pic X that have $C^2 = C \cdot K_X = -1$.

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- In general, we call curves corresponding to such classes -1 *curves* or *lines*.

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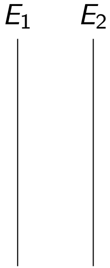
  the strict transform of
- lines through two of the points;
- conics through five of the points;
- cubics through seven of the points, singular at one of them;
- quartics through eight of the points, singular at three of them;
- quintics through eight of the points, singular at six of them;
- sextics through eight of the points, singular at all of them, containing one of them as a triple point.

Degree 7

Blow up 2 points



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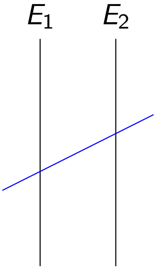

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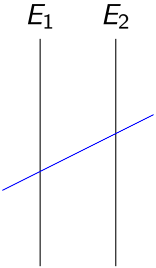
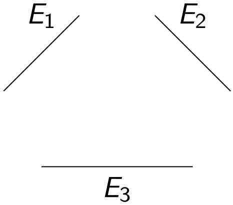


Degree 6

Blow up 3 points



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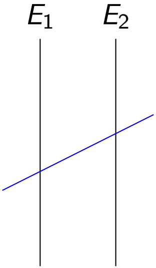

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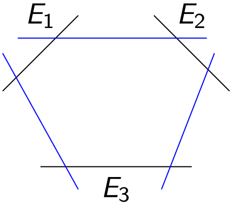

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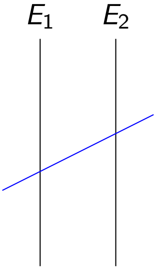

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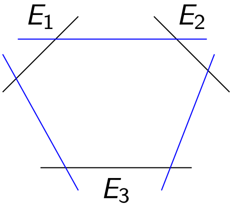
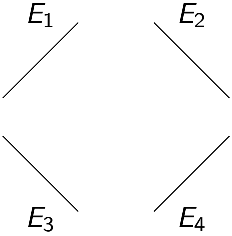


Degree 5

Blow up 4 points



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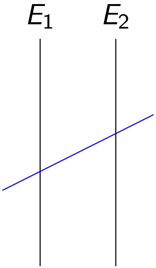

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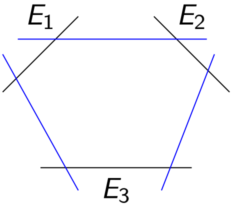

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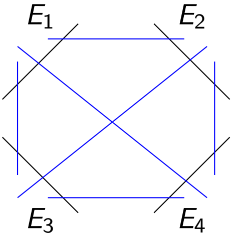

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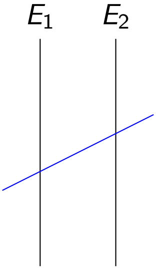

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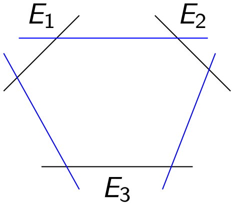

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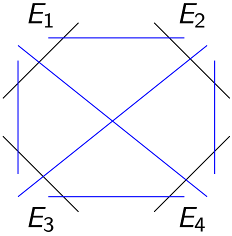

d	1	2	3	4	5	6	7	8
lines on X	240	56	27	16	10	6	3	1

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We blow up 6 points. So the 27 lines are:
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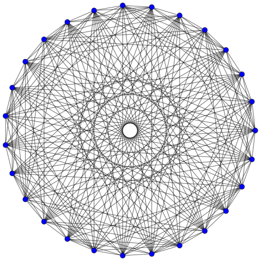
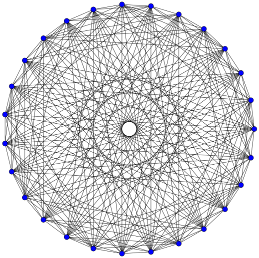


The intersection graph of the lines is the complement of the Schläfli graph.



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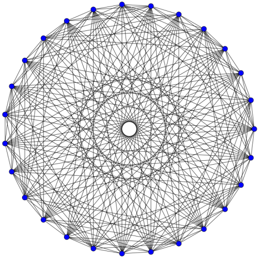

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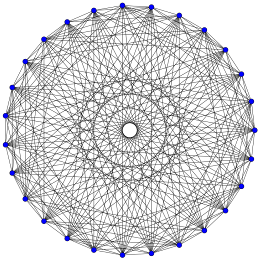

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If a point is contained in n lines, then the lines form a full subgraph (*clique*) of size n.

$\Longrightarrow$ maximal size of cliques gives an upper bound for the number of lines through one point.

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$\implies$ The maximal size of a clique is three; the upper bound given by the graph is sharp!

We also saw that there are at most 45 Eckardt points on a cubic surface; we can see this from the graph as well. $\frac{27 \cdot 5}{3} = 45$.

Degree two

Del Pezzo surfaces of degree two are double covers of $\mathbb{P}^2$ that are ramified over a smooth quartic curve. They have 56 'lines'.

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<u>Fact</u>

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Point in four lines: *generalized Eckardt point*. Generalized Eckardt points are always outside the ramification curve.

Degree one

To get a del Pezzo surface X of degree one we blow up the plane in 8 points $P_1, \ldots, P_8$ in general position. We obtain the following 'lines':

- 8 lines above the P_i
- $\binom{8}{2} = 28$ lines through 2 of the P_i
- $\binom{8}{5} = 56$ conics through 5 of the P_i
- $7 \cdot \binom{8}{7} = 56$ cubics through 7 of the P_i with a singular point at one of them
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We find a total of 240 lines on X!

How can we study the configurations of these 240 lines?

The root system E_8

Consider the lattice in $\mathbb{R}^8$ given by

$$\Lambda = \left\{ (x_i) \in \mathbb{Z}^8 \cup \left(\mathbb{Z} + \tfrac{1}{2}\mathbb{Z}\right)^8 \mid \sum x_i \in 2\mathbb{Z} \right\}.$$

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Fact

The 240 lines on a del Pezzo surface of degree one are isomorphic to the root system E_8.

$\{-1 \text{ curves on } X\} \longrightarrow K_X^{\perp},\ e \longmapsto e + K_X$

Symmetries in the graph on the 240 lines

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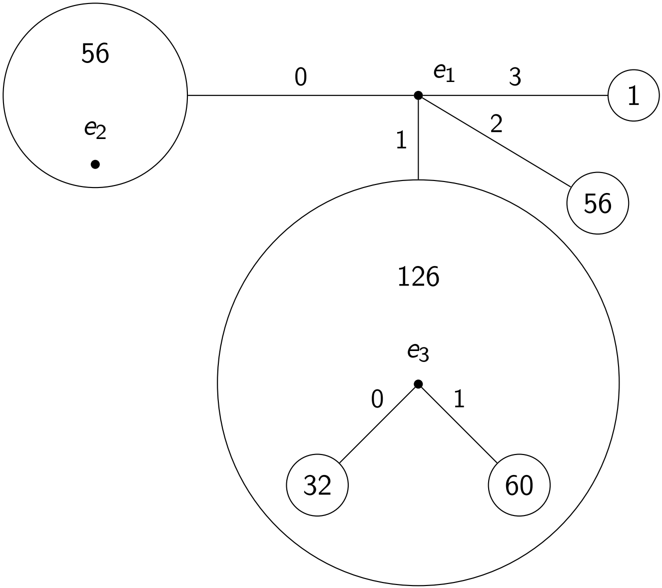
- To study the different cliques in G we use this symmetry.

The graph G on the 240 lines



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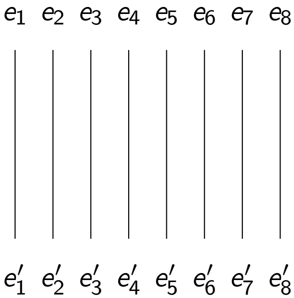
Del Pezzo surfaces of degree one are double covers of a cone in $\mathbb{P}^3$, ramified over a smooth sextic curve.

with edges of weight three $\longleftrightarrow$ points on the ramification curve

no edges of weight three $\longleftrightarrow$ points outside the ramification curve

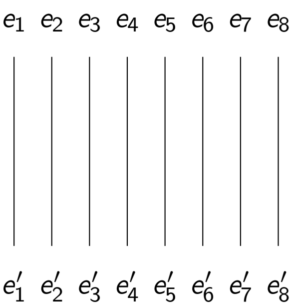
Maximal cliques in G

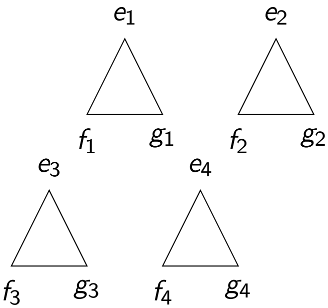


$$e_i \cdot e_i' = 3$$

Cliques with edges of weight 3: maximal size 16. There are 2025 such cliques.

Maximal cliques in G





$e_i \cdot e_i' = 3$

$e_i \cdot f_i = e_i \cdot g_i = f_i \cdot g_i = 2$

Cliques without edges of weight 3: maximal size 12. There are 179200 such cliques.

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I turns out that for a DP1, the upper bound given by the graph is (almost) never sharp, making this case different from all other degrees.

Classical geometry - the case of cliques of size 16

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Assume that the four lines L_1, L_2, L_3 and L_4 all intersect in one point P. Then the three cubics $C_{7,8}$, $C_{8,7}$, and $C_{6,5}$ do not all go through P.

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Corollary

No six pairs of lines intersecting with multiplicity three go through one point, hence a point on the ramification curve on a del Pezzo surface of degree 1 lies on at most ten lines in characteristic $\neq 2$.

Actual statement of the theorem

Del Pezzo surfaces of degree one are double covers of a cone in $\mathbb{P}^3$, ramified over a smooth sextic curve.

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Let X be a del Pezzo surface of degree one over an algebraically closed field k.
Any point on the ramification curve is contained in at most 16 lines for $\mathrm{char} k = 2$, and in at most 10 lines for $\mathrm{char} k \neq 2$.
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The upper bounds are sharp in all characteristics, except possibly in characteristic 5 outside the ramification curve.

Example with 16 lines in characteristic 2

Let $f = x^5 + x^2 + 1 \in \mathbb{F}_2[x]$, and let $F \cong \mathbb{F}_2(\alpha)$ where α is a root of f. Define the following eight points in $\mathbb{P}^2_F$.

$Q_1 = (0 : 1 : 1)$	$Q_3 = (1 : 0 : 1)$	$Q_5 = (1 : 1 : 1)$	$Q_7 = (\alpha^{24} : \alpha^{25} : 1)$
$Q_2 = (0 : 1 : \alpha^{19})$	$Q_4 = (1 : 0 : \alpha^5)$	$Q_6 = (\alpha^{20} : \alpha^{20} : \alpha^{16})$	$Q_8 = (\alpha^{30} : 1 : \alpha^5)$

The blow-up of $\mathbb{P}^2_F$ in $(Q_1, \ldots, Q_8)$ is a del Pezzo surface S.

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These sixteen curves all go through the point $(0 : 0 : 1) \in \mathbb{P}^2_F$, so they are concurrent on S.

Example with 10 lines in characteristic 0

Define the following eight points in $\mathbb{P}^2_{\mathbb{Q}}$.

$$Q_1 = (0 : 1 : 1); \qquad Q_5 = (1 : 1 : 1);$$
$$Q_2 = (0 : 5 : 3); \qquad Q_6 = (4 : 4 : 5);$$
$$Q_3 = (1 : 0 : 1); \qquad Q_7 = (-2 : 2 : 1);$$
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These ten curves all go through the point $(0:0:1) \in \mathbb{P}^2$, so they are concurrent on S.

Thank you!