Chapter 1

Preliminaries

1.1 Topological spaces

1.1.1 The notion of topological space

The topology on a set X is usually defined by specifying its open subsets of X. However, in dealing with topological vector spaces, it is often more convenient to define a topology by specifying what the neighbourhoods of each point are.

Definition 1.1.1. *A* topology τ *on a set* X *is a family of subsets of* X *which satisfies the following conditions:*
(O1) *the empty set* $\emptyset$ *and the whole* X *are both in* τ
(O2) τ *is closed under finite intersections*
(O3) τ *is closed under arbitrary unions*
The pair (X, τ) *is called a* topological space*.*

The sets $O \in \tau$ are called *open sets* of X and their complements $C = X \setminus O$ are called *closed sets* of X. A subset of X may be neither closed nor open, either closed or open, or both. A set that is both closed and open is called a *clopen set*.

Definition 1.1.2. *Let* (X, τ) *be a topological space.*

- *A subfamily* $\mathcal{B}$ *of* τ *is called a* basis *if every open set can be written as a union of sets in* $\mathcal{B}$*.*
- *A subfamily* $\mathcal{X}$ *of* τ *is called a* subbasis *if the finite intersections of its sets form a basis, i.e. every open set can be written as a union of finite intersections of sets in* $\mathcal{X}$*.*

Therefore, a topology τ on X is completely determined by a basis or a subbasis.

Example 1.1.3. *Let $\mathcal{S}$ be the collection of all semi-infinite intervals of the real line of the forms $(-\infty, a)$ and $(a, +\infty)$, where $a \in \mathbb{R}$. $\mathcal{S}$ is not a base for any topology on $\mathbb{R}$. To show this, suppose it were. Then, for example, $(-\infty, 1)$ and $(0, \infty)$ would be in the topology generated by S, being unions of a single base element, and so their intersection $(0, 1)$ would be by the axiom (O2) of topology. But $(0, 1)$ clearly cannot be written as a union of elements in $\mathcal{S}$.*

Proposition 1.1.4. *Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. $\mathcal{B}$ is a basis for a topology τ on X iff the following hold:*

1. *$\mathcal{B}$ covers X, i.e. $\forall\, x \in X$, $\exists\, B \in \mathcal{B}$ s.t. $x \in B$.*
2. *If $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then $\exists\, B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subseteq B_1 \cap B_2$.*

Proof. (Sheet 1, Exercise 1) □

Definition 1.1.5. *Let (X, τ) be a topological space and $x \in X$. A subset U of X is called a* neighbourhood *of x if it contains an open set containing the point x, i.e. $\exists\, O \in \tau$ s.t. $x \in O \subseteq U$. The family of all neighbourhoods of a point $x \in X$ is denoted by $\mathcal{F}(x)$.*

In order to define a topology on a set by the family of neighbourhoods of each of its points, it is convenient to introduce the notion of filter. Note that the notion of filter is given on a set which does not need to carry any other structure. Thus this notion is perfectly independent of the topology.

Definition 1.1.6. *A* filter *on a set X is a family $\mathcal{F}$ of subsets of X which fulfills the following conditions:*

(F1) *the empty set $\emptyset$ does not belong to $\mathcal{F}$*
(F2) *$\mathcal{F}$ is closed under finite intersections*
(F3) *any subset of X containing a set in $\mathcal{F}$ belongs to $\mathcal{F}$*

Definition 1.1.7. *A family $\mathcal{B}$ of subsets of X is called a* basis of a filter *$\mathcal{F}$ if*

1. *$\mathcal{B} \subseteq \mathcal{F}$*
2. *$\forall A \in \mathcal{F}$, $\exists\, B \in \mathcal{B}$ s.t. $B \subseteq A$*

Examples 1.1.8.

a) *The family $\mathcal{G}$ of all subsets of a set X containing a fixed non-empty subset A is a filter and $\mathcal{B} = \{A\}$ is its base. $\mathcal{G}$ is called the* principle filter *generated by A.*
b) *Given a topological space X and $x \in X$, the family $\mathcal{F}(x)$ is a filter.*
c) *Let $S := \{x_n\}_{n \in \mathbb{N}}$ be a sequence of points in a set X. Then the family $\mathcal{F} := \{A \subset X : |S \setminus A| < \infty\}$ is a filter and it is known as the* filter associated *to S. For each $m \in \mathbb{N}$, set $S_m := \{x_n \in S : n \geq m\}$. Then $\mathcal{B} := \{S_m : m \in \mathbb{N}\}$ is a basis for $\mathcal{F}$.*

Proof. (Sheet 1, Exercise 2). □

Theorem 1.1.9. *Given a topological space X and a point $x \in X$, the filter of neighbourhoods $\mathcal{F}(x)$ satisfies the following properties.*
(N1) *For any $A \in \mathcal{F}(x)$, $x \in A$.*
(N2) *For any $A \in \mathcal{F}(x)$, $\exists\, B \in \mathcal{F}(x)$: $\forall\, y \in B$, $A \in \mathcal{F}(y)$.*
Viceversa, if for each point x in a set X we are given a filter $\mathcal{F}_x$ fulfilling the properties (N1) and (N2) then there exists a unique topology τ s.t. for each $x \in X$, $\mathcal{F}_x$ is the family of neighbourhoods of x, i.e. $\mathcal{F}_x \equiv \mathcal{F}(x)$, $\forall x \in X$.

This means that a topology on a set is uniquely determined by the family of neighbourhoods of each of its points.

Proof.
$\Rightarrow$ Let (X, τ) be a topological space, $x \in X$ and $\mathcal{F}(x)$ the filter of neighbourhoods of x. Then (N1) trivially holds by definition of neighbourhood of x. To show (N2), let us take $A \in \mathcal{F}(x)$. Since A is a neighbourhood of x, there exists $B \in \tau$ s.t. $x \in B \subseteq A$. Then clearly $B \in \mathcal{F}(x)$. Moreover, since for any $y \in B$ we have that $y \in B \subseteq A$ and B is open, we can conclude that $A \in \mathcal{F}(y)$.
$\Leftarrow$ Assume that for any $x \in X$ we have a filter $\mathcal{F}_x$ fulfilling (N1) and (N2). Let us define $\tau := \{O \subseteq X : \text{ if } x \in O \text{ then } O \in \mathcal{F}_x\}$. Since each $\mathcal{F}_x$ is a filter, τ is a topology. Indeed:

- $\emptyset \in \tau$ by definition of τ. Also $X \in \tau$, because for any $x \in X$ and any $A \in \mathcal{F}_x$ we clearly have $X \supseteq A$ and so by (F3) $X \in \mathcal{F}_x$.
- By (F2) we have that τ is closed under finite intersection.
- Let U be an arbitrary union of sets $U_i \in \tau$ and let $x \in U$. Then there exists at least one i s.t. $x \in U_i$ and so $U_i \in \mathcal{F}_x$ because $U_i \in \tau$. But $U \supseteq U_i$, then by (F3) we get that $U \in \mathcal{F}_x$ and so $U \in \tau$.

It remains to show that τ on X is actually s.t. $\mathcal{F}_x \equiv \mathcal{F}(x)$, $\forall x \in X$.

- Any $U \in \mathcal{F}(x)$ is a neighbourhood of x and so there exists $O \in \tau$ s.t. $x \in O \subseteq U$. Then, by definition of τ, we have $O \in \mathcal{F}_x$ and so (F3) implies that $U \in \mathcal{F}_x$. Hence, $\mathcal{F}(x) \subseteq \mathcal{F}_x$.
- Let $U \in \mathcal{F}_x$ and set $W := \{y \in U : U \in \mathcal{F}_y\} \subseteq U$. Since $x \in U$ by (N1), we also have $x \in W$. Moreover, if $y \in W$ then by (N2) there exists $V \in \mathcal{F}_y$ s.t. $\forall z \in V$ we have $U \in \mathcal{F}_z$. This means that $z \in W$ and so $V \subseteq W$. Then $W \in \mathcal{F}_y$ by (F3). Hence, we have showed that if $y \in W$ then $W \in \mathcal{F}_y$, i.e. $W \in \tau$. Summing up, we have just constructed an open set W s.t. $x \in W \subseteq U$, i.e. $U \in \mathcal{F}(x)$, and so $\mathcal{F}_x \subseteq \mathcal{F}(x)$.

□

Definition 1.1.10. *Given a topological space X, a basis $\mathcal{B}(x)$ of the filter of neighbourhoods $\mathcal{F}(x)$ of a point $x \in X$ is called a* base of neighbourhoods of x, *i.e. $\mathcal{B}(x)$ is a subcollection of $\mathcal{F}(x)$ s.t. every neighbourhood in $\mathcal{F}(x)$ contains* *one in $\mathcal{B}(x)$. The elements of $\mathcal{B}(x)$ are called* basic neighbourhoods of x. *If a base of neighbourhoods is given for each point $x \in X$, we speak of* base of neighbourhoods of X.

Example 1.1.11. *The open sets of a topological space other than the empty set always form a base of neighbourhoods.*

Theorem 1.1.12. *Given a topological space X and a point $x \in X$, a base of open neighbourhoods $\mathcal{B}(x)$ satisfies the following properties.*
(B1) *For any $U \in \mathcal{B}(x)$, $x \in U$.*
(B2) *For any $U_1, U_2 \in \mathcal{B}(x)$, $\exists U_3 \in \mathcal{B}(x)$ s.t. $U_3 \subseteq U_1 \cap U_2$.*
(B3) *If $y \in U \in \mathcal{B}(x)$, then $\exists W \in \mathcal{B}(y)$ s.t. $W \subseteq U$.*
Viceversa, if for each point x in a set X we are given a collection of subsets $\mathcal{B}_x$ fulfilling the properties (B1), (B2) and (B3) then there exists a unique topology τ s.t. for each $x \in X$, $\mathcal{B}_x$ is a base of neighbourhoods of x, i.e. $\mathcal{B}_x \equiv \mathcal{B}(x)$, $\forall x \in X$.

Proof. The proof easily follows by using Theorem 1.1.9. □

The previous theorem gives a further way of introducing a topology on a set. Indeed, starting from a base of neighbourhoods of X, we can define a topology on X by setting that a set is open iff whenever it contains a point it also contains a basic neighbourhood of the point. Thus a topology on a set X is uniquely determined by a base of neighbourhoods of each of its points.

1.1.2 Comparison of topologies

Any set X may carry several different topologies. When we deal with topological vector spaces, we will very often encounter this situation of a set, in fact a vector space, carrying several topologies (all compatible with the linear structure, in a sense that is going to be specified soon). In this case, it is convenient being able to compare topologies.

Definition 1.1.13. *Let τ, τ' be two topologies on the same set X. We say that τ is* coarser *(or* weaker*) than τ', in symbols $\tau \subseteq \tau'$, if every subset of X which is open for τ is also open for τ', or equivalently, if every neighborhood of a point in X w.r.t. τ is also a neighborhood of that same point in the topology τ'. In this case τ' is said to be* finer *(or* stronger*) than τ'.*

Denote by $\mathcal{F}(x)$ and $\mathcal{F}'(x)$ the filter of neighbourhoods of a point $x \in X$ w.r.t. τ and w.r.t. τ', respectively. Then: τ is coarser than τ' iff for any point $x \in X$ we have $\mathcal{F}(x) \subseteq \mathcal{F}'(x)$ (this means that every subset of X which belongs to $\mathcal{F}(x)$ also belongs to $\mathcal{F}'(x)$).

Two topologies τ and τ' on the same set X *coincide* when they give the same open sets or the same closed sets or the same neighbourhoods of each point; equivalently, when τ is both coarser and finer than τ'. Two basis of neighbourhoods of a set are *equivalent* when they define the same topology.

Remark 1.1.14. *Given two topologies on the same set, it may very well happen that none is finer than the other. If it is possible to establish which one is finer, then we say that the two topologies are* comparable.

Example 1.1.15.
The cofinite topology τ_c *on* $\mathbb{R}$*, i.e.* $\tau_c := \{U \subseteq \mathbb{R} : U = \emptyset \text{ or } \mathbb{R} \setminus U \text{ is finite}\}$*, and the topology* τ_i *having* $\{(-\infty, a) : a \in \mathbb{R}\}$ *as a basis are incomparable. In fact, it is easy to see that* $\tau_i = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ *as these are the unions of sets in the given basis. In particular, we have that* $\mathbb{R} - \{0\}$ *is in* τ_c *but not* τ_i*. Moreover, we have that* $(-\infty, 0)$ *is in* τ_i *but not* τ_c*. Hence,* τ_c *and* τ_i *are incomparable.*

It is always possible to construct at least two topologies on every set X by choosing the collection of open sets to be as large as possible or as small as possible:

- the *trivial topology*: every point of X has only one neighbourhood which is X itself. Equivalently, the only open subsets are $\emptyset$ and X. The only possible basis for the trivial topology is $\{X\}$.
- the *discrete topology*: given any point $x \in X$, every subset of X containing x is a neighbourhood of x. Equivalently, every subset of X is open (actually clopen). In particular, the singleton $\{x\}$ is a neighbourhood of x and actually is a basis of neighbourhoods of x. The collection of all singletons is a basis for the discrete topology.

Note that the discrete topology on a set X is finer than any other topology on X, while the trivial topology is coarser than all the others. Topologies on a set form thus a partially ordered set, having a maximal and a minimal element, respectively the discrete and the trivial topology.

A useful criterion to compare topologies on the same set is the following:

Theorem 1.1.16 (Hausdorff's criterion)**.**
For each $x \in X$*, let* $\mathcal{B}(x)$ *a base of neighbourhoods of* x *for a topology* τ *on* X *and* $\mathcal{B}'(x)$ *a base of neighbourhoods of* x *for a topology* τ' *on* X*.*
$\tau \subseteq \tau'$ *iff* $\forall x \in X, \forall U \in \mathcal{B}(x)\ \exists V \in \mathcal{B}'(x)$ *s.t.* $x \in V \subseteq U$*.*

The Hausdorff criterion could be paraphrased by saying that smaller neighborhoods make larger topologies. This is a very intuitive theorem, because the smaller the neighbourhoods are the easier it is for a set to contain neighbourhoods of all its points and so the more open sets there will be.

Proof.

$\Rightarrow$ Suppose $\tau \subseteq \tau'$. Fixed any point $x \in X$, let $U \in \mathcal{B}(x)$. Then, since U is a neighbourhood of x in (X, τ), there exists $O \in \tau$ s.t. $x \in O \subseteq U$. But $O \in \tau$ implies by our assumption that $O \in \tau'$, so U is also a neighbourhood of x in (X, τ'). Hence, by Definition 1.1.10 for $\mathcal{B}'(x)$, there exists $V \in \mathcal{B}'(x)$ s.t. $V \subseteq U$.

$\Leftarrow$ Conversely, let $W \in \tau$. Then for each $x \in W$, since $\mathcal{B}(x)$ is a base of neighbourhoods w.r.t. τ, there exists $U \in \mathcal{B}(x)$ such that $x \in U \subseteq W$. Hence, by assumption, there exists $V \in \mathcal{B}'(x)$ s.t. $x \in V \subseteq U \subseteq W$. Then $W \in \tau'$. $\square$

1.1.3 Reminder of some simple topological concepts

Definition 1.1.17. *Given a topological space (X, τ) and a subset S of X, the* subset or induced topology *on S is defined by $\tau_S := \{S \cap U \mid U \in \tau\}$. That is, a subset of S is open in the subset topology if and only if it is the intersection of S with an open set in (X, τ).*
Alternatively, we can define the subspace topology for a subset S of X as the coarsest topology for which the inclusion map $\iota : S \hookrightarrow X$ is continuous.

Note that (S, τ_s) is a topological space in its own.

Definition 1.1.18. *Given a collection of topological space (X_i, τ_i), where $i \in I$ (I is an index set possibly uncountable), the* product topology *on the Cartesian product $X := \prod_{i \in I} X_i$ is defined in the following way: a set U is open in X iff it is an arbitrary union of sets of the form $\prod_{i \in I} U_i$, where each $U_i \in \tau_i$ and $U_i \neq X_i$ for only finitely many i.*
Alternatively, we can define the product topology to be the coarsest topology for which all the canonical projections $p_i : X \to X_i$ are continuous.

Given a topological space X, we define:

Definition 1.1.19.

- *The* closure *of a subset $A \subseteq X$ is the smallest closed set containing A. It will be denoted by $\bar{A}$. Equivalently, $\bar{A}$ is the intersection of all closed subsets of X containing A.*
- *The* interior *of a subset $A \subseteq X$ is the largest open set contained in it. It will be denoted by $\mathring{A}$. Equivalently, $\mathring{A}$ is the union of all open subsets of X contained in A.*