2. *Every family of seminorms on a vector space containing a norm induces a Hausdorff locally convex topology.*
3. *Given an open subset Ω of $\mathbb{R}^d$ with the euclidean topology, the space $\mathcal{C}(\Omega)$ of real valued continuous functions on Ω with the so-called* topology of uniform convergence on compact sets *is a locally convex t.v.s.. This topology is defined by the family $\mathcal{P}$ of all the seminorms on $\mathcal{C}(\Omega)$ given by*
$$p_K(f) := \max_{x\in K} |f(x)|, \forall K \subset \Omega \text{ compact} .$$
Moreover, $(\mathcal{C}(\Omega), \tau_{\mathcal{P}})$ is Hausdorff, because the family $\mathcal{P}$ is clearly separating. In fact, if $p_K(f) = 0$, $\forall K$ compact subsets of Ω then in particular $p_{\{x\}}(f) = |f(x)| = 0\ \forall x \in \Omega$, which implies $f \equiv 0$ on Ω.
More generally, for any X locally compact we have that $\mathcal{C}(X)$ with the topology of uniform convergence on compact subsets of X is a locally convex Hausdorff t.v.s.

To introduce two other examples of l.c. Hausdorff t.v.s. we need to recall some standard general notations. Let $\mathbb{N}_0$ be the set of all non-negative integers. For any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ one defines $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. For any $\beta \in \mathbb{N}_0^d$, the symbol D^β denotes the partial derivative of order $|\beta|$ where $|\beta| := \sum_{i=1}^d \beta_i$, i.e.
$$D^\beta := \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}} = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}}.$$

Examples 4.3.5.

1. *Let $\Omega \subseteq \mathbb{R}^d$ open in the euclidean topology. For any $k \in \mathbb{N}_0$, let $\mathcal{C}^k(\Omega)$ be the set of all real valued $k-$*times continuously differentiable functions *on Ω, i.e. all the derivatives of f of order $\leq k$ exist (at every point of Ω) and are continuous functions in Ω. Clearly, when $k = 0$ we get the set $\mathcal{C}(\Omega)$ of all real valued continuous functions on Ω and when $k = \infty$ we get the so-called set of all* infinitely differentiable functions *or* smooth functions *on Ω. For any $k \in \mathbb{N}_0$, $\mathcal{C}^k(\Omega)$ (with pointwise addition and scalar multiplication) is a vector space over $\mathbb{R}$. The topology given by the following family of seminorms on $\mathcal{C}^k(\Omega)$:*
$$p_{m,K}(f) := \sup_{\substack{\beta\in\mathbb{N}_0^d \\ |\beta|\leq m}} \sup_{x\in K} \left|(D^\beta f)(x)\right|, \forall K \subseteq \Omega \text{ compact}, \forall m \in \{0, 1, \dots, k\},$$
makes $\mathcal{C}^k(\Omega)$ into a locally convex Hausdorff t.v.s.. (Note that when $k = \infty$ we have $m \in \mathbb{N}_0$.)

2. *The* Schwartz space *or* space of rapidly decreasing functions *on* $\mathbb{R}^d$ *is defined as the set* $\mathcal{S}(\mathbb{R}^d)$ *of all real-valued functions which are defined and infinitely differentiable on* $\mathbb{R}^d$ *and which have the additional property (regulating their growth at infinity) that all their derivatives tend to zero at infinity faster than any inverse power of* x*, i.e.*

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} \left| x^\alpha D^\beta f(x) \right| < \infty, \ \forall \alpha, \beta \in \mathbb{N}_0^d \right\}.$$

(For example, any smooth function f *with compact support in* $\mathbb{R}^d$ *is in* $\mathcal{S}(\mathbb{R}^d)$*, since any derivative of* f *is continuous and supported on a compact subset of* $\mathbb{R}^d$*, so* $x^\alpha(D^\beta f(x))$ *has a maximum in* $\mathbb{R}^d$ *by the extreme value theorem.)*

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ *is a vector space over* $\mathbb{R}$ *and the topology given by the family* $\mathcal{Q}$ *of seminorms on* $\mathcal{S}(\mathbb{R}^d)$*:*

$$q_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^d} \left| x^\alpha D^\beta f(x) \right|, \ \forall \alpha, \beta \in \mathbb{N}_0^d$$

makes $\mathcal{S}(\mathbb{R}^d)$ *into a locally convex Hausdorff t.v.s.. Indeed, the family is clearly separating, because if* $q_{\alpha,\beta}(f) = 0$, $\forall \alpha, \beta \in \mathbb{N}_0^d$ *then in particular* $q_{o,o}(f) = \sup_{x\in\mathbb{R}^d} |f(x)| = 0 \ \forall x \in \mathbb{R}^d$*, which implies* $f \equiv 0$ *on* $\mathbb{R}^d$*.*
Note that $\mathcal{S}(\mathbb{R}^d)$ *is a linear subspace of* $\mathcal{C}^\infty(\mathbb{R}^d)$*, but its topology* $\tau_\mathcal{Q}$ *on* $\mathcal{S}(\mathbb{R}^d)$ *is finer than the subspace topology induced on it by* $\mathcal{C}^\infty(\mathbb{R}^d)$*. (Sheet 10, Exercise 1)*

4.4 The finest locally convex topology

In the previous sections we have seen how to generate topologies on a vector space which makes it into a locally convex t.v.s.. Among all of them, there is the finest one (i.e. the one having the largest number of open sets) which is usually called the *finest locally convex topology* on the given vector space.

Proposition 4.4.1. *The finest locally convex topology on a vector space* X *is the topology induced by the family of all seminorms on* X *and it is a Hausdorff topology.*

Proof.
Let us denote by $\mathcal{S}$ the family of all seminorms on the vector space X. By Theorem 4.2.9, we know that the topology $\tau_\mathcal{S}$ induced by $\mathcal{S}$ makes X into a locally convex t.v.s. We claim that $\tau_\mathcal{S}$ is the finest locally convex topology. In

fact, if there was a finer locally convex topology τ (i.e. if $\tau_{\mathcal{S}} \subseteq \tau$ with (X, τ) locally convex t.v.s.) then Theorem 4.2.9 would give that τ is also induced by a family $\mathcal{P}$ of seminorms. But surely $\mathcal{P} \subseteq \mathcal{S}$ and so $\tau = \tau_{\mathcal{P}} \subseteq \tau_{\mathcal{S}}$ by definition of induced topology. Hence, $\tau = \tau_{\mathcal{S}}$.

It remains to show that $(X, \tau_{\mathcal{S}})$ is Hausdorff. By Lemma 4.3.2, it is enough to prove that $\mathcal{S}$ is separating. Let $x \in X \setminus \{o\}$ and let $\mathcal{B}$ be an algebraic basis of the vector space X containing x. Define the linear functional $L : X \to \mathbb{R}$ as $L(x) = 1$ and $L(y) = 0$ for all $y \in \mathcal{B} \setminus \{x\}$. Then it is easy to see that $s := |L|$ is a seminorm, so $s \in \mathcal{S}$ and $s(x) \neq 0$, which proves that $\mathcal{S}$ is separating. □

An alternative way of describing the finest locally convex topology on a vector space X without using the seminorms is the following:

Proposition 4.4.2. *The collection of all absorbing absolutely convex sets of a vector space X is a basis of neighbourhoods of the origin for the finest locally convex topology on X.*

Proof. Let τ_{max} be the finest locally convex topology on X and $\mathcal{A}$ the collection of all absorbing absolutely convex sets of X. By Theorem 4.1.14, we know that every locally convex t.v.s. has a basis of neighbourhood of the origin consisting of absorbing absolutely convex subsets of X. Then clearly the basis of neighbourhoods of the origin $\mathcal{B}_{max}$ of τ_{max} is contained in $\mathcal{A}$. Hence, $\tau_{max} \subseteq \tau$ where τ denote the topology generated by $\mathcal{A}$. On the other hand, $\mathcal{A}$ fulfills all the properties required in Theorem 4.1.14 and so τ also makes X into a locally convex t.v.s.. Hence, by definition of finest locally convex topology, $\tau \subseteq \tau_{max}$. □

This result can be clearly proved also using the Proposition 4.4.1 and the correspondence between Minkowski functionals and absorbing absolutely convex subsets of X introduced in the Section 4.2.

Proposition 4.4.3. *Every linear functional on a vector space X is continuous w.r.t. the finest locally convex topology on X.*

Proof. Let $L : X \to \mathbb{K}$ be a linear functional on a vector space X. For any $\varepsilon > 0$, we denote by $B_\varepsilon(0)$ the open ball in $\mathbb{K}$ of radius ε and center $0 \in \mathbb{K}$, i.e. $B_\varepsilon(0) := \{k \in \mathbb{K} : |k| < \varepsilon\}$. Then we have that $L^{-1}(B_\varepsilon(0)) = \{x \in X : |L(x)| < \varepsilon\}$. It is easy to verify that the latter is an absorbing absolutely convex subset of X and so, by Proposition 4.4.2, it is a neighbourhood of the origin in the finest locally convex topology on X. Hence L is continuous at the origin and so, by Proposition 2.1.15-3), L is continuous everywhere in X. □

4.5 Direct limit topology on a countable dimensional t.v.s.

In this section we are going to give an important example of finest locally convex topology on an infinite dimensional vector space, namely the *direct limit topology* on any countable dimensional vector space. For simplicity, we are going to focus on $\mathbb{R}-$vector spaces.

Definition 4.5.1. *Let X be an infinite dimensional vector space whose dimension is countable. The* direct limit topology *(or* finite topology*) τ_f on X is defined as follows:*
$U \subseteq X$ is open in τ_f iff $U \cap W$ is open in the euclidean topology on W, $\forall\, W \subset X$ with $dim(W) < \infty$.
Equivalently, if we fix a basis $\{x_n\}_{n\in\mathbb{N}}$ of X and if for any $n \in \mathbb{N}$ we set $X_n := span\{x_1, \ldots, x_n\}$ s.t. $X = \bigcup_{i=1}^{\infty} X_i$ and $X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$, then
$U \subseteq X$ is open in τ_f iff $U \cap X_i$ is open in the euclidean topology on V_i for every $i \in \mathbb{N}$.

Theorem 4.5.2. *Let X be an infinite dimensional vector space whose dimension is countable endowed with the finite topology τ_f. Then:*
a) (X, τ_f) is a Hausdorff locally convex t.v.s.
b) τ_f is the finest locally convex topology on X

Proof.
a) We leave to the reader the proof of the fact that τ_f is compatible with the linear structure of X (Sheet 10, Exercise 3) and we focus instead on proving that τ_f is a locally convex topology. To this aim we are going to show that for any open neighbourhood U of the origin in (X, τ_f) there exists an open convex neighbourhood $U' \subseteq U$.

Let $\{x_i\}_{i\in\mathbb{N}}$ be an $\mathbb{R}$-basis for X and set $X_n := span\{x_1, \ldots, x_n\}$ for any $n \in \mathbb{N}$. We proceed (by induction on $n \in \mathbb{N}$) to construct an increasing sequence $C_n \subseteq U \cap X_n$ of convex subsets as follows:

- For $n = 1$: Since $U \cap X_1$ is open in $X_1 = \mathbb{R}x_1$, we have that there exists $a_1 \in \mathbb{R}, a_1 > 0$ such that $C_1 := \left\{\lambda_1 x_1 \mid -a_1 \leq \lambda_1 \leq a_1\right\} \subseteq U \cap X_1$.
- Inductive assumption on n: We assume we have found $a_1, \ldots, a_n \in \mathbb{R}_+$ such that $C_n := \left\{\lambda_1 x_1 + \ldots + \lambda_n x_n \mid -a_i \leq \lambda_i \leq a_i \,; i \in \{1, \ldots, n\}\right\} \subseteq U \cap X_n$. Note that C_n is closed (in X_n, as well as) in X_{n+1}.
- For $n + 1$: We claim $\exists\ a_{n+1} > 0, a_{n+1} \in \mathbb{R}$ such that
  $C_{n+1} := \left\{\lambda_1 x_1 + \ldots + \lambda_n x_n + \lambda_{n+1} x_{n+1} | -a_i \leq \lambda_i \leq a_i \,; i \in \{1, \ldots, n+1\}\right\} \subseteq U \cap X_{n+1}$.
  Proof of claim: If the claim does not hold, then $\forall\ N \in \mathbb{N}\ \exists\ x^N \in X_{n+1}$ s.t.
  $$x^N = \lambda_1^N x_1 + \ldots \lambda_n^N x_n + \lambda_{n+1}^N x_{n+1}$$

with $-a_i \leq \lambda_i^N \leq a_i$ for $i \in \{1, \ldots, n\}$, $-\frac{1}{N} \leq \lambda_{n+1}^N \leq \frac{1}{N}$ and $x^N \notin U$.

But x^N has form $x^N = \underbrace{\lambda_1^N x_1 + \ldots + \lambda_n^N x_n}_{\in C_n} + \lambda_{n+1}^N x_{n+1}$, so $\{x^N\}_{N \in \mathbb{N}}$ is a bounded sequence in $X_{n+1} \setminus U$. Therefore, we can find a subsequence $\{x^{N_j}\}_{j \in \mathbb{N}}$ which is convergent as $j \to \infty$ to $x \in C_n \subseteq U$ (since C_n is closed in X_{n+1} and the $n+1-$th component of x^{N_j} tends to 0 as $j \to \infty$). Hence, the sequence $\{x^{N_j}\}_{j \in \mathbb{N}} \subseteq X_{n+1} \setminus U$ converges to $x \in U$ but this contradicts the fact that $X_{n+1} \setminus U$ is closed in X_{n+1}. This establishes the claim.

Now for any $n \in \mathbb{N}$ consider

$$D_n := \left\{ \lambda_1 x_1 + \ldots + \lambda_n x_n \mid -a_i < \lambda_i < a_i \; ; i \in \{1, \ldots, n\} \right\},$$

then $D_n \subset C_n \subseteq U \cap X_n$ is open and convex in X_n. Then $U' := \cup_{n \in \mathbb{N}} D_n$ is an open and convex neighbourhood of the origin in (X, τ_f) and $U' \subseteq U$.

b) Let us finally show that τ_f is actually the finest locally convex topology τ_{max} on X. Since we have already showed that τ_f is a l.c. topology on X, clearly we have $\tau_f \subseteq \tau_{max}$ by definition of finest l.c. topology on X.

Conversely, let us consider $U \subseteq X$ open in τ_{max}. We want to show that U is open in τ_f, i.e. $W \cap U$ is open in the euclidean topology on W for any finite dimensional subspace W of X. Now each W inherits τ_{max} from X. Let us denote by τ_{max}^W the subspace topology induced by (X, τ_{max}) on W. By definition of subspace topology, we have that $W \cap U$ is open in τ_{max}^W. Moreover, by Proposition 4.4.1, we know that (X, τ_{max}) is a Hausdorff t.v.s. and so (W, τ_{max}^W) is a finite dimensional Hausdorff t.v.s. (see by Proposition 2.1.15-1). Therefore, τ_{max}^W has to coincide with the euclidean topology by Theorem 3.1.1 and, consequently, $W \cap U$ is open w.r.t. the euclidean topology on W. □

We actually already know a concrete example of countable dimensional space with the finite topology:

Example 4.5.3. *Let $n \in \mathbb{N}$ and $\underline{x} = (x_1, \ldots, x_n)$. Denote by $\mathbb{R}[\underline{x}]$ the space of polynomials in the n variables $x_1, \ldots, x_n$ with real coefficients and by*

$$\mathbb{R}_d[\underline{x}] := \{ f \in \mathbb{R}[\underline{x}] | \deg f \leq d \}, d \in \mathbb{N}_0,$$

then $\mathbb{R}[\underline{x}] := \bigcup_{d=0}^{\infty} \mathbb{R}_d[\underline{x}]$. The finite topology τ_f on $\mathbb{R}[\underline{x}]$ is then given by: $U \subseteq \mathbb{R}[\underline{x}]$ is open in τ_f iff $\forall d \in \mathbb{N}_0$, $U \cap \mathbb{R}_d[\underline{x}]$ is open in $\mathbb{R}_d[\underline{x}]$ with the euclidean topology.