4.6 Continuity of linear mappings on locally convex spaces

Since locally convex spaces are a particular class of topological vector spaces, the natural functions to be considered on this spaces are continuous linear maps. In this section, we present a necessary and sufficient condition for the continuity of a linear map between two l.c. spaces, bearing only on the seminorms inducing the two topologies.

For simplicity, let us start with linear functionals on a l.c. space. Recall that for us $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ endowed with the euclidean topology given by the absolute value $|\cdot|$. In this section, for any $\varepsilon > 0$ we denote by $B_\varepsilon(0)$ the open ball in $\mathbb{K}$ of radius ε and center $0 \in \mathbb{K}$ i.e. $B_\varepsilon(0) := \{k \in \mathbb{K} : |k| < \varepsilon\}$.

Proposition 4.6.1. *Let τ be a locally convex topology on a vector space X generated by a directed family $\mathcal{Q}$ of seminorms on X. Then $L : X \to \mathbb{K}$ is a τ-continuous linear functional iff there exists $q \in \mathcal{Q}$ such that L is q-continuous, i.e.*

$$\exists q \in \mathcal{Q}, \exists C > 0 \;\; s.t. \, |L(x)| \leq Cq(x), \, \forall \, x \in X. \tag{4.7}$$

Proof.
Let us first observe that since X and $\mathbb{K}$ are both t.v.s. by Proposition 2.1.15-3) the continuity of L is equivalent to its continuity at the origin. Therefore, it is enough to prove the criterion for the continuity of L at the origin.

($\Rightarrow$) Suppose that L is τ-continuous at the origin in X. Then we have that $L^{-1}(B_1(0)) = \{x \in X : |L(x)| < 1\}$ is an open neighbourhood of the origin in (X, τ). Since the family $\mathcal{Q}$ inducing τ is directed, a basis of neighbourhood of the origin in (X, τ) is given by $\mathcal{B}_d$ as in (4.5). Therefore, $\exists B \in \mathcal{B}_d$ s.t. $B \subseteq L^{-1}(B_1(0))$, i.e.

$$\exists \; q \in \mathcal{Q}, \, \exists \; r > 0 \;\text{ s.t. }\; r\mathring{U}_q \subseteq L^{-1}(B_1(0)). \tag{4.8}$$

Then for any $\varepsilon > 0$ we get $r\varepsilon\mathring{U}_q \subseteq \varepsilon L^{-1}(B_1(0)) = L^{-1}(B_\varepsilon(0))$. This proves that L is $q-$continuous at the origin, because $r\varepsilon\mathring{U}_q$ is clearly an open neighbourhood of the origin in X w.r.t. the topology generated by the single seminorm q.

($\Leftarrow$) Suppose that there exists $q \in \mathcal{Q}$ s.t. L is q-continuous in X. Then, since τ is the topology induced by the whole family $\mathcal{Q}$ which is finer than the one induced by the single seminorm q, we clearly have that L is also $\tau-$continuous.

Note that by simply observing that $|L|$ is a seminorm and by using Proposition 4.2.10 we get that (4.7) is equivalent to (4.8) and so to the q-continuity of L at the origin.

□

By using this result together with Proposition 4.2.14 we get the following.

Corollary 4.6.2. *Let τ be a locally convex topology on a vector space X generated by a family $\mathcal{P} := \{p_i\}_{i\in I}$ of seminorms on X. Then $L : X \to \mathbb{K}$ is a τ-continuous linear functional iff there exist $n \in \mathbb{N}$, $i_1, \ldots, i_n \in I$ such that L is $\left(\max_{k=1,\ldots,n} p_{i_k}\right)$-continuous, i.e.*

$$\exists\, n \in \mathbb{N}, \exists\, i_1, \ldots, i_n \in I, \exists\, C > 0 \text{ s.t. } |L(x)| \leq C \max_{k=1,\ldots,n} p_{i_k}(x),\ \forall\, x \in X.$$

The proof of Proposition 4.6.1 can be easily modified to get the following more general criterion for the continuity of any linear map between two locally convex spaces.

Theorem 4.6.3. *Let X and Y be two locally convex t.v.s. whose topologies are respectively generated by the families $\mathcal{P}$ and $\mathcal{Q}$ of seminorms on X. Then $f : X \to Y$ linear is continuous iff*

$$\forall\, q \in \mathcal{Q}, \exists\, n \in \mathbb{N}, \exists\, p_1, \ldots, p_n \in \mathcal{P}, \exists\, C > 0 \,:\, q(f(x)) \leq C \max_{i=1,\ldots,n} p_i(x),\ \forall\, x \in X.$$

Proof. (Exercise) □

Chapter 5

The Hahn-Banach Theorem and its applications

5.1 The Hahn-Banach Theorem

One of the most important results in the theory of t.v.s. is the Hahn-Banach theorem (HBT). It is named for Hans Hahn and Stefan Banach who proved this theorem independently in the late 1920s, dealing with the problem of extending continuous linear functionals defined on a subspace of a seminormed vector space to the whole space. We will see that actually this extension problem can be reduced to the problem of separating by a closed hyperplane a convex open set and an affine submanifold (the image by a translation of a linear subspace) which do not intersect. Indeed, there are two main versions of HBT showing respectively the analytic and the geometric side of this result.

Before stating these two versions of HBT, let us recall the notion of hyperplane in a vector space (we always consider vector spaces over the field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$). A *hyperplane* H in a vector space X over $\mathbb{K}$ is a maximal proper linear subspace of X or, equivalently, a linear subspace of codimension one, i.e. $dim X/H = 1$. Another equivalent formulation is that a hyperplane is a set of the form $\varphi^{-1}(\{0\})$ for some linear functional $\varphi : X \to \mathbb{K}$ not identically zero. The translation by a non-null vector of a hyperplane will be called *affine hyperplane*.

Theorem 5.1.1 (Analytic form of Hahn-Banach theorem).
Let p be a seminorm on a vector space X over $\mathbb{K}$, M a linear subspace of X, and f a linear functional on M such that

$$|f(x)| \leq p(x), \forall\, x \in M. \tag{5.1}$$

There exists a linear functional $\tilde{f}$ on X extending f, i.e. $\tilde{f}(x) = f(x) \,\forall\, x \in M$, and such that

$$|\tilde{f}(x)| \leq p(x), \forall\, x \in X. \tag{5.2}$$

Theorem 5.1.2 (Geometric form of Hahn-Banach theorem).
Let X be a topological vector space over $\mathbb{K}$, N a linear subspace of X, and Ω a convex open subset of X such that $N \cap \Omega = \emptyset$. Then there exists a closed hyperplane H of X such that

$$N \subseteq H \quad \textit{and} \quad H \cap \Omega = \emptyset. \tag{5.3}$$

It should be remarked that the vector space X does not apparently carry any topology in Theorem 5.1.1, but actually the datum of a seminorm on X is equivalent to the datum of the topology induced by this seminorm. It is then clear that the conditions (5.1) and (5.2) imply the $p-$continuity of the functions f and $\tilde{f}$, respectively.

Let us also stress the fact that in Theorem 5.1.2 neither local convexity nor the Hausdorff separation property are assumed on the t.v.s. X. Moreover, it is easy to see that the geometric form of HBT could have been stated also in an affine setting, namely starting with any affine submanifold N of X which does not intersect the open convex subset Ω and getting a closed affine hyperplane fulfilling (5.3).

We first prove Theorem 5.1.2 and then show how to derive from this the analytic form Theorem 5.1.1.

Before starting the proof, let us fix one more definition. A *cone* C in a vector space X over $\mathbb{R}$ is a subset of X which is closed under addition and multiplication by positive scalars.

Proof. Theorem 5.1.2
We assume that $\Omega \neq \emptyset$, otherwise there is nothing to prove.

1) Existence of a linear subspace H of X maximal for (5.3).
This first part of the proof is quite simple and consists in a straightforward application of Zorns lemma. In fact, consider the family $\mathcal{F}$ of all the linear subspaces L of X such that

$$N \subseteq L \quad \text{and} \quad L \cap \Omega = \emptyset. \tag{5.4}$$

$\mathcal{F}$ is clearly non-empty since N belongs to it by assumption. If we take now a totally ordered subfamily $\mathcal{C}$ of $\mathcal{F}$ (totally ordered for the inclusion relation $\subseteq$), then the union of all the linear subspaces belonging to $\mathcal{C}$ is a linear subspace of X having the properties in (5.4). Hence, we can apply Zorn's lemma applies and conclude that there exists at least a maximal element H in $\mathcal{F}$.

2) H is closed in X.
The fact that H and Ω do not intersect gives that H is contained in the

complement of Ω in X. This implies that also its closure $\overline{H}$ does not intersect Ω. Indeed, since Ω is open, we get

$$\overline{H} \subseteq \overline{X \setminus \Omega} = X \setminus \Omega.$$

Then $\overline{H}$ is a linear subspace (as closure of a linear subspace) of X, which is disjoint from Ω and which contains H and so N, i.e. $\overline{H} \in \mathcal{F}$. Hence, as H is maximal in $\mathcal{F}$, it must coincide with its closure. Note that the fact that H is closed guarantees that the quotient space X/H is a Hausdorff t.v.s. (see Proposition 2.3.5).

3) H is an hyperplane

We want to show that H is a hyperplane, i.e. that $dim(X/H) = 1$. To this aim we distinguish the two cases when $\mathbb{K} = \mathbb{R}$ and when $\mathbb{K} = \mathbb{C}$.

3.1) Case $\mathbb{K} = \mathbb{R}$

Let $\phi : X \to X/H$ be the canonical map. Since ϕ is an open linear mapping (see Proposition 2.3.2), $\phi(\Omega)$ is an open convex subset of X/H. Also we have that $\phi(\Omega)$ does not contain the origin $\hat{o}$ of X/H. Indeed, if $\hat{o} \in \phi(\Omega)$ holds, then there would exist $x \in \Omega$ s.t. $\phi(x) = \hat{o}$ and so $x \in H$, which would contradict the assumption $H \cap \Omega = \emptyset$. Let us set:

$$A = \bigcup_{\lambda > 0} \lambda \phi(\Omega).$$

Then the subset A of X/H is open, convex and it is a cone which does not contain the origin $\hat{o}$.

Let us observe that X/H has at least dimension 1. If $dim(X/H) = 0$ then $X/H = \{\hat{o}\}$ and so $X = H$ which contradicts the fact that Ω does not intersect H (recall that we assumed Ω is non-empty). Suppose that $dim(X/H) \geq 2$, then to get our conclusion it will suffice to show the following claims:

Claim 1: The boundary ∂A of A must contain at least one point $x \neq \hat{o}$.

Claim 2: The point $-x$ cannot belong to A.

In fact, once Claim 1 is established, we have that $x \notin A$, because $x \in \partial A$ and A is open. This together with Claim 2 gives that both x and $-x$ belong to the complement of A in X/H and, therefore, so does the straight line L defined by these two points. (If there was a point $y \in L \cap A$ then any positive multiple of y would belong to $L \cap A$, as A is a cone. Hence, for some $\lambda > 0$ we would have $x = \lambda y \in L \cap A$, which contradicts the fact that $x \notin A$.) Then:

- $\phi^{-1}(L)$ is a linear subspace of X
- $\phi^{-1}(L) \cap \Omega = \emptyset$, since $L \cap A = \emptyset$
- $\phi^{-1}(L) \supsetneq H$ because $\hat{o} = \phi(H) \subseteq L$ but $L \neq \{\hat{o}\}$ since $x \neq \hat{o}$ is in L.

This contradicts the maximality of H and so $dim(X/H) = 1$.

To complete the proof of 3.1) let us show the two claims.

Proof. of Claim 1
Suppose that $\partial A = \{\hat{o}\}$. This means that A has empty boundary in the set $(X/H) \setminus \{\hat{o}\}$ and so that A is a connected component of $(X/H) \setminus \{\hat{o}\}$. Since $dim(X/H) \geq 2$, the space $(X/H) \setminus \{\hat{o}\}$ is arc-connected and so it is itself a connected space. Hence, $A = (X/H) \setminus \{\hat{o}\}$ which contradicts the convexity of A since $(X/H) \setminus \{\hat{o}\}$ is clearly not convex. □

Proof. of Claim 2
Suppose $-x \in A$. Then, as A is open, there is a neighborhood V of $-x$ entirely contained in A. This implies that $-V$ is a neighborhood of x. Since x is a boundary point of A, there exists $y \in (-V) \cap A$. But then $-y \in V \subset A$ and so, by the convexity of A, the whole line segment between y and $-y$ is contained in A, in particular $\hat{o}$, which contradicts the definition of A. □

3.2) Case $\mathbb{K} = \mathbb{C}$
Although here we are consider the scalars to be the complex numbers, we may view X as a vector space over the real numbers and it is obvious that its topology, as originally given, is still compatible with its linear structure. By step 3.1) above, we know that there exists a real hyperplane H_0 of X which contains N and does not intersect Ω. By a real hyperplane, we mean that H_0 is a linear subspace of X viewed as a vector space over the field of real numbers such that $dim_{\mathbb{R}}(X/H) = 1$.

Now it is easy to see that $iN = N$ (here $i = \sqrt{-1}$). Hence, setting $H := H_0 \cap iH_0$, we have that $N \subseteq H$ and $H \cap \Omega = \emptyset$. Then to complete the proof it remains to show that this H is a complex hyperplane. It is obviously a complex linear subspace of X and its real codimension is ≥ 1 and ≤ 2 (since the intersection of two distinct hyperplanes is always a linear subspace with codimension two). Hence, its complex codimension is equal to one.

□

Proof. Theorem 5.1.1
Let p be a seminorm on the vector space X, M a linear subspace of X, and f a linear functional defined on M fulfilling (5.1). As already remarked before, this means that f is continuous on M w.r.t. the topology induced by p on X (which makes X a l.c. t.v.s.).

Consider the subset $N := \{x \in M : f(x) = 1\}$. Taking any vector $x_0 \in N$, it is easy to see that $N - x_0 = Ker(f)$ (i.e. the kernel of f in M), which is a hyperplane of M and so a linear subspace of X. Therefore, setting $M_0 := N - x_0$, we have the following decomposition of M:

$$M = M_0 \oplus \mathbb{K}x_0,$$

where $\mathbb{K}x_0$ is the one-dimensional linear subspace through x_0. In other words

$$\forall\, x \in M,\ \exists!\, \lambda \in \mathbb{K}, y \in M_0 :\ x = y + \lambda x_0.$$

Then

$$\forall\, x \in M,\ f(x) = f(y) + \lambda f(x_0) = \lambda f(x_0) = \lambda,$$

which means that the values of f on M are completely determined by the ones on N. Consider now the open unit semiball of p:

$$U := \mathring{U}_p = \{x \in X : p(x) < 1\},$$

which we know being an open convex subset of X endowed with the topology induced by p. Then $N \cap U = \emptyset$ because if there was $x \in N \cap U$ then $p(x) < 1$ and $f(x) = 1$, which contradict (5.1).

By Theorem 5.1.2 (affine version), there exists a closed affine hyperplane H of X with the property that $N \subseteq H$ and $H \cap U = \emptyset$. Then $H - x_0$ is a hyperplane and so the kernel of a continuous linear functional $\tilde{f}$ on X non-identically zero.

Arguing as before (consider here the decomposition $X = (H - x_0) \oplus \mathbb{K}x_0$), we can deduce that the values of $\tilde{f}$ on X are completely determined by the ones on N and so on H (because for any $h \in H$ we have $h - x_0 \in Ker(\tilde{f})$ and so $\tilde{f}(h) - \tilde{f}(x_0) = \tilde{f}(h - x_0) = 0$). Since $\tilde{f} \not\equiv 0$, we have that $\tilde{f}(x_0) \neq 0$ and w.l.o.g. we can assume $\tilde{f}(x_0) = 1$ i.e. $\tilde{f} \equiv 1$ on H. Therefore, for any $x \in M$ there exist unique $\lambda \in \mathbb{K}$ and $y \in N - x_0 \subseteq H - x_0$ s.t. $x = y + \lambda x_0$, we get that:

$$\tilde{f}(x) = \lambda \tilde{f}(x_0) = \lambda = \lambda f(x_0) = f(x),$$

i.e. f is the restriction of $\tilde{f}$ to M. Furthermore, the fact that $H \cap U = \emptyset$ means that $\tilde{f}(x) = 1$ implies $p(x) \geq 1$. Then for any $y \in X$ s.t. $\tilde{f}(y) \neq 0$ we have that: $\tilde{f}\left(\frac{y}{\tilde{f}(y)}\right) = 1$ and so that $p\left(\frac{y}{\tilde{f}(y)}\right) \geq 1$ which implies that $|\tilde{f}(y)| \leq p(y)$. The latter obviously holds for $\tilde{f}(y) = 0$. Hence, (5.2) is established. □

5.2 Applications of Hahn-Banach theorem

The Hahn-Banach theorem is frequently applied in analysis, algebra and geometry, as will be seen in the forthcoming course. We will briefly indicate in this section some applications of this theorem to problems of separation of convex sets and to the multivariate moment problem. From now on we will focus on t.v.s. over the field of real numbers.