where $\mathbb{K}x_0$ is the one-dimensional linear subspace through x_0. In other words

$$\forall\, x \in M,\ \exists!\, \lambda \in \mathbb{K}, y \in M_0 :\ x = y + \lambda x_0.$$

Then

$$\forall\, x \in M,\ f(x) = f(y) + \lambda f(x_0) = \lambda f(x_0) = \lambda,$$

which means that the values of f on M are completely determined by the ones on N. Consider now the open unit semiball of p:

$$U := \mathring{U}_p = \{x \in X : p(x) < 1\},$$

which we know being an open convex subset of X endowed with the topology induced by p. Then $N \cap U = \emptyset$ because if there was $x \in N \cap U$ then $p(x) < 1$ and $f(x) = 1$, which contradict (5.1).

By Theorem 5.1.2 (affine version), there exists a closed affine hyperplane H of X with the property that $N \subseteq H$ and $H \cap U = \emptyset$. Then $H - x_0$ is a hyperplane and so the kernel of a continuous linear functional $\tilde{f}$ on X non-identically zero.

Arguing as before (consider here the decomposition $X = (H - x_0) \oplus \mathbb{K}x_0$), we can deduce that the values of $\tilde{f}$ on X are completely determined by the ones on N and so on H (because for any $h \in H$ we have $h - x_0 \in Ker(\tilde{f})$ and so $\tilde{f}(h) - \tilde{f}(x_0) = \tilde{f}(h - x_0) = 0$). Since $\tilde{f} \not\equiv 0$, we have that $\tilde{f}(x_0) \neq 0$ and w.l.o.g. we can assume $\tilde{f}(x_0) = 1$ i.e. $\tilde{f} \equiv 1$ on H. Therefore, for any $x \in M$ there exist unique $\lambda \in \mathbb{K}$ and $y \in N - x_0 \subseteq H - x_0$ s.t. $x = y + \lambda x_0$, we get that:

$$\tilde{f}(x) = \lambda \tilde{f}(x_0) = \lambda = \lambda f(x_0) = f(x),$$

i.e. f is the restriction of $\tilde{f}$ to M. Furthermore, the fact that $H \cap U = \emptyset$ means that $\tilde{f}(x) = 1$ implies $p(x) \geq 1$. Then for any $y \in X$ s.t. $\tilde{f}(y) \neq 0$ we have that: $\tilde{f}\left(\frac{y}{\tilde{f}(y)}\right) = 1$ and so that $p\left(\frac{y}{\tilde{f}(y)}\right) \geq 1$ which implies that $|\tilde{f}(y)| \leq p(y)$. The latter obviously holds for $\tilde{f}(y) = 0$. Hence, (5.2) is established. □

5.2 Applications of Hahn-Banach theorem

The Hahn-Banach theorem is frequently applied in analysis, algebra and geometry, as will be seen in the forthcoming course. We will briefly indicate in this section some applications of this theorem to problems of separation of convex sets and to the multivariate moment problem. From now on we will focus on t.v.s. over the field of real numbers.

5.2.1 Separation of convex subsets of a real t.v.s.

Let X t.v.s. over the field of real numbers and H be a closed affine hyperplane of X. We say that two disjoint subsets A and B of X are *separated* by H if A is contained in one of the two closed half-spaces determined by H and B is contained in the other one. We can express this property in terms of functionals. Indeed, since $H = L^{-1}(\{a\})$ for some $L : X \to \mathbb{R}$ linear not identically zero and some $a \in \mathbb{R}$, we can write that A and B are separated by H if and only if:

$$\exists\, a \in \mathbb{R} \text{ s.t. } L(A) \geq a \text{ and } L(B) \leq a.$$

where for any $S \subseteq X$ the notation $L(S) \leq a$ simply means $\forall s \in S, L(s) \leq a$ (and analogously for $\geq, <, >, =, \neq$).
We say that A and B are *strictly separated* by H if at least one of the two inequalities is strict. (Note that there are several definition in literature for the strict separation but for us it will be just the one defined above) In the present subsection we would like to investigate whether one can separate, or strictly separate, two disjoint convex subsets of a real t.v.s..

Proposition 5.2.1. *Let X be a t.v.s. over the real numbers and A, B two disjoint convex subsets of X.*

a) If A is open nonempty and B is nonempty, then there exists a closed affine hyperplane H of X separating A and B, i.e. there exists $a \in \mathbb{R}$ and a functional $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq a$ and $L(B) \leq a$.

b) If in addition B is open, the hyperplane H can be chosen so as to strictly separate A and B, i.e. there exists $a \in \mathbb{R}$ and $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq a$ and $L(B) < a$.

c) If A is a cone and B is open, then a can be chosen to be zero, i.e. there exists $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq 0$ and $L(B) < 0$.

Proof.

a) Consider the set $A - B := \{a - b : a \in A, b \in B\}$. Then: $A - B$ is an open subset of X as it is the union of the open sets $A - y$ as y varies over B; $A - B$ is convex as it is the Minkowski sum of the convex sets A and $-B$; and $o \notin (A - B)$ because if this was the case then there would be at least a point in the intersection of A and B which contradicts the assumption that they are disjoint. By applying Theorem 5.1.2 to $N = \{o\}$ and $U = A - B$ we have that there is a closed hyperplane H of X which does not intersect $A - B$ (and passes through the origin) or, which is equivalent, there exists a linear form f on X not identically zero such that $f(A-B) \neq 0$. Then there exists a linear form L on X not identically zero such that $L(A-B) > 0$ (in the case $f(A-B) < 0$ just take $L := -f$) i.e.

$$\forall\, x \in A, \forall\, y \in B, \;\; L(x) > L(y). \tag{5.5}$$

Since $B \neq \emptyset$ we have that $a := \inf_{x \in A} L(x) > -\infty$. Then (5.5) implies that $L(B) \leq a$ and we clearly have $L(A) \geq a$.

b) Let now both A and B be open convex and nonempty disjoint subsets of X. By part a) we have that there exists $a \in \mathbb{R}$ and $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq a$ and $L(B) \leq a$. Suppose that there exists $b \in B$ s.t. $L(b) = a$. Since B is open, for any $x \in X$ there exists $\varepsilon > 0$ s.t. for all $t \in [0, \varepsilon]$ we have $b + tx \in B$. Therefore, as $L(B) \leq a$, we have that

$$L(b + tx) \leq a, \forall\, t \in [0, \varepsilon]. \tag{5.6}$$

Now fix $x \in X$, consider the function $f(t) := L(b + tx)$ for all $t \in \mathbb{R}$ whose first derivative is clearly given by $f'(t) = L(x)$ for all $t \in \mathbb{R}$. Then (5.6) means that $t = 0$ is a point of local minimum for f and so $f'(0) = 0$ i.e. $L(x) = 0$. As x is an arbitrary point of x, we get $L \equiv 0$ on X which is a contradiction. Hence, $L(B) < a$.

c) Let now A be a nonempty convex cone of X and B an open convex nonempty subset of X s.t. $A \cap B = \emptyset$. By part a) we have that there exists $a \in \mathbb{R}$ and $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq a$ and $L(B) \leq a$. Since A is a cone, for any $t > 0$ we have that $tA \subseteq A$ and so $tL(A) = L(tA) \geq a$ i.e. $L(A) \geq \frac{a}{t}$. This implies that $L(A) \geq \inf_{t>0} \frac{a}{t} = 0$. Moreover, part a) also gives that $L(B) < L(A)$. Therefore, for any $t > 0$ and any $x \in A$, we have in particular $L(B) < L(tx) = tL(x)$ and so $L(B) \leq \inf_{t>0} tL(x) = 0$. Since B is also open, we can exactly proceed as in part b) to get $L(B) < 0$.

□

Let us show now two interesting consequences of this result which we will use in the following subsection.

Corollary 5.2.2. *Let X be a vector space over $\mathbb{R}$ endowed with the finest locally convex topology φ. If C is a nonempty closed cone in X and $x_0 \in X \setminus C$ then there exists a linear functional $L : X \to \mathbb{R}$ non identically zero s.t. $L(C) \geq 0$ and $L(x_0) < 0$.*

Proof. As C is closed in (X, φ) and $x_0 \in X \setminus C$, we have that $X \setminus C$ is an open neighbourhood of x_0. Then the local convexity of (X, φ) guarantees that there exists an open convex neighbourhood V of x_0 s.t. $V \subseteq X \setminus C$ i.e. $V \cap C = \emptyset$. By Proposition 5.2.1-c), we have that there exists $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(\mathring{C}) \geq 0$ and $L(V) < 0$, in particular $L(x_0) < 0$. $\square$

Before giving the second corollary, let us introduce some notations. Given a cone C in a t.v.s. (X, τ) we define the first and the second dual of C w.r.t. τ respectively as follows:

$$C_\tau^\vee := \{\ell : X \to \mathbb{R} \text{ linear} \,|\, \ell \text{ is } \tau-\text{continuous and } \ell(C) \geq 0\}$$

$$C_\tau^{\vee\vee} := \{x \in X \,|\, \forall\, \ell \in C_\tau^\vee,\, \ell(x) \geq 0\}.$$

Corollary 5.2.3. *Let X be a vector space over $\mathbb{R}$ endowed with the finest locally convex topology φ. If C is a nonempty cone in X, then $\overline{C}^\varphi = C_\varphi^{\vee\vee}$.*

Proof. Let us first observe that $C \subseteq C_\varphi^{\vee\vee}$, because for any $x \in C$ and any $\ell \in C_\varphi^\vee$ we have by definition of first dual of C that $\ell(x) \geq 0$ and so that $x \in C_\varphi^{\vee\vee}$. Then we get that $\overline{C}^\varphi \subseteq \overline{C_\varphi^{\vee\vee}}^\varphi$. But $C_\varphi^{\vee\vee}$ is closed since $C_\varphi^{\vee\vee} = \bigcap_{\ell \in C_\varphi^\vee} \ell([0, +\infty))$ and each $\ell \in C_\varphi^\vee$ is φ-continuous. Hence, $\overline{C}^\varphi \subseteq C_\varphi^{\vee\vee}$.

Conversely, suppose there exists $x_0 \in C_\varphi^{\vee\vee} \setminus \overline{C}^\varphi$. By Corollary 5.2.2, there exists a linear functional $L : X \to \mathbb{R}$ non identically zero s.t. $L(\overline{C}^\varphi) \geq 0$ and $L(x_0) < 0$. As $L(C) \geq 0$ and every linear functional is $\varphi-$continuous, we have $L \in C_\varphi^\vee$. This together with the fact that $L(x_0) < 0$ give $x_0 \notin C_\varphi^{\vee\vee}$, which is a contradiction. Hence, $\overline{C}^\varphi = C_\varphi^{\vee\vee}$. $\square$

5.2.2 Multivariate real moment problem

Let $d \in \mathbb{N}$ and let $\mathbb{R}[\underline{x}]$ be the ring of polynomials with real coefficients and d variables $\underline{x} := (x_1, \ldots, x_d)$. Fixed a subset K of $\mathbb{R}^d$, we denote by

$$Psd(K) := \{p \in \mathbb{R}[\underline{x}] : p(\underline{x}) \geq 0, \forall x \in K\}.$$

Definition 5.2.4 (Multivariate real $K-$moment problem)**.**
Given a closed subset K of $\mathbb{R}^d$ and a linear functional $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$, does there exists a nonnegative finite Borel measure μ s.t.

$$L(p) = \int_{\mathbb{R}^d} p(\underline{x})\mu(d\underline{x}),\ \forall p \in \mathbb{R}[\underline{x}]$$

and $supp(\mu) \subseteq K$ (where $supp(\mu)$ denotes the support of the measure μ)?

*If such a measure exists, we say that μ is a K-*representing measure *for L and that it is a solution to the $K-$moment problem for L.*

A necessary condition for the existence of a solution to the $K-$moment problem for the linear functional L is clearly that L is nonnegative on $Psd(K)$. In fact, if there exists a representing measure μ for L then for all $p \in Psd(K)$ we have

$$L(p) = \int_{\mathbb{R}^d} p((\underline{x}))\mu(d\underline{x}) = \int_K p((\underline{x}))\mu(d\underline{x}) \geq 0$$

since μ is nonnegative and supported on K and p is nonnegative on K.

It is then natural to ask if the nonnegative of L on $Psd(K)$ is also sufficient. The answer is positive and it was established by Riesz in 1923 for $d = 1$ and by Haviland for any $d \geq 2$.

Theorem 5.2.5 (Riesz-Haviland Theorem)**.** *Let K be a closed subset of $\mathbb{R}^d$ and $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ be linear. L has a $K-$representing measure if and only if $L(Psd(K)) \geq 0$.*

Note that this theorem provides a complete solution for the $K-$ moment problem but it is quite unpractical! In fact, it reduces the solvability of the $K-$moment problem to the problem of classifying all polynomials which are nonnegative on a prescribed closed subset K of $\mathbb{R}^d$ i.e. to characterize $Psd(K)$. This is actually a hard problem to be solved for general K and it is a core question in real algebraic geometry. For example, if we think of the case $K = \mathbb{R}^d$ then for $d = 1$ we know that $Psd(K) = \sum \mathbb{R}[\underline{x}]^2$, where $\sum \mathbb{R}[\underline{x}]^2$ denotes the set of squares of polynomials. However, for $d \geq 2$ this equality does not hold anymore as it was proved by Hilbert in 1888. It is now clear that to make the conditions of the Riesz-Haviland theorem actually checkable we need to be able to write/approximate a non-negative polynomial on K in a way that makes its non-negativity apparent, i.e. as a sum of squares or as an element of quadratic modules of $\mathbb{R}[\underline{x}]$. For a special class of closed subsets of $\mathbb{R}^d$ we actually have such representations and we can get better conditions than the one of Riesz-Haviland type to solve the $K-$moment problem.

Definition 5.2.6. *Given a finite set of polynomials $S := \{g_1, \ldots, g_s\}$, we call the* basic closed semialgebraic set *generated by S the following*

$$K_S := \{\underline{x} \in \mathbb{R}^d : g_i(\underline{x}) \geq 0,\ i = 1, \ldots, s\}.$$

Definition 5.2.7. *A subset M of $\mathbb{R}[\underline{x}]$ is said to be a* quadratic module *if $1 \in M$, $M + M \subseteq M$ and $h^2 M \subseteq M$ for any $h \in \mathbb{R}[\underline{x}]$.*

Note that each quadratic module is a cone in $\mathbb{R}[\underline{x}]$.

Definition 5.2.8. *A quadratic module M of $\mathbb{R}[\underline{x}]$ is called* Archimedean *if there exists $N \in \mathbb{N}$ s.t. $N - (\sum_{i=1}^d x_i^2) \in M$.*

For $S := \{g_1, \ldots, g_s\}$ finite subset of $\mathbb{R}[\underline{x}]$, we define the *quadratic module generated by S* to be:

$$M_S := \left\{ \sum_{i=0}^{s} \sigma_i g_i : \sigma_i \in \sum \mathbb{R}[\underline{x}]^2, i = 0, 1, \ldots, s \right\},$$

where $g_0 := 1$.

Remark 5.2.9. *Note that $M_S \subseteq Psd(K_S)$ and M_S is the smallest quadratic module of $\mathbb{R}[\underline{x}]$ containing S.*

Consider now the finite topology on $\mathbb{R}[\underline{x}]$ (see Definition 4.5.1) which we have proved to be the finest locally convex topology on this space (see Proposition 4.5.2) and which we therefore denote by φ. By Corollary 5.2.3, we get that

$$\overline{M_S}^{\varphi} = (M_S)^{\vee\vee}_{\varphi} \tag{5.7}$$

Moreover, the *Putinar Positivstellesatz* (1993), a milestone result in real algebraic geometry, provides that if M_S is Archimedean then

$$Psd(K_S) \subseteq \overline{M_S}^{\varphi}. \tag{5.8}$$

Note that M_S is Archimedean implies that K_S is compact while the converse is in general not true (see e.g. M. Marshall, Positive polynomials and sum of squares, 2008).

Combining (5.7) and (5.8), we get the following result.

Proposition 5.2.10. *Let $S := \{g_1, \ldots, g_s\}$ be a finite subset of $\mathbb{R}[\underline{x}]$ and $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ linear. Assume that M_S is Archimedean. Then there exists a K_S-representing measure μ for L if and only if $L(M_S) \geq 0$, i.e. $L(h^2 g_i) \geq 0$ for all $h \in \mathbb{R}[\underline{x}]$ and for all $i \in \{1, \ldots, s\}$.*

Proof. Suppose that $L(M_S) \geq 0$ and let us consider the finite topology φ on $\mathbb{R}[\underline{x}]$. Then the linear functional L is φ-continuous and so $L \in (M_S)^{\vee}_{\varphi}$. Moreover, as M_S is assumed to be Archimedean we have

$$Psd(K_S) \overset{(5.8)}{\subseteq} \overline{M_S}^{\varphi} \overset{(5.7)}{=} (M_S)^{\vee\vee}_{\varphi}.$$

Since any $p \in Psd(K_S)$ is also an element of $(M_S)^{\vee\vee}_{\varphi}$, we have that for any $\ell \in (M_S)^{\vee}_{\varphi}$, $\ell(Psd(K_S)) \geq 0$ and in particular $L(Psd(K_S)) \geq 0$. Hence, by Riesz-Haviland theorem we get the existence of a K_S-representing measure μ for L.

Conversely, suppose that the there exists a K_S-representing measure μ for L. Then for all $p \in M_S$ we have in particular that

$$L(p) = \int_{\mathbb{R}^d} p(\underline{x})\mu(d\underline{x})$$

which is nonnegative as μ is a nonnegative measure supported on K_S and $p \in M_S \subseteq Psd(K_S)$. $\square$

From this result and its proof we understand that whenever we know that $Psd(K_S) \subseteq \overline{M_S}^{\varphi}$, we need to check only that $L(M_S) \geq 0$ to find out whether there exists a solution for the K_S-moment problem for L. Then it makes sense to look for closure results of this kind in the case when M_S is not Archimedean and so we cannot apply the Putinar Positivstellesatz. Actually whenever we know that $Psd(K_S) \subseteq \overline{M_S}^{\tau}$ where τ is a locally convex topology on $\mathbb{R}[\underline{x}]$, the condition $L(M_S) \geq 0$ is necessary and sufficient for the existence of a solution of the K_S-moment problem for any $\tau-$continuous functional on $\mathbb{R}[\underline{x}]$ (see M. Ghasemi, S. Kuhlmann, E. Samei, 2012). This relationship between the closure of quadratic modules and the representability of functionals continuous w.r.t. locally convex topologies started a new research line in the study of the moment problem which is still bringing interesting results.