Proposition 2.1.13.
a) If B is a balanced subset of a t.v.s. X then so is $\bar{B}$.
b) If B is a balanced subset of a t.v.s. X and $o \in \mathring{B}$ then $\mathring{B}$ is balanced.

Proof. (Sheet 3, Exercise 2 b) c)) □

Proof. of Theorem 2.1.10.
<u>*Necessity part.*</u>
Suppose that X is a t.v.s. then we aim to show that the filter of neighbourhoods of the origin $\mathcal{F}$ satisfies the properties 1,2,3,4,5. Let $U \in \mathcal{F}$.

1. obvious, since every set $U \in \mathcal{F}$ is a neighbourhood of the origin o.
2. Since by the definition of t.v.s. the addition $(x, y) \mapsto x+y$ is a continuous mapping, the preimage of U under this map must be a neighbourhood of $(o, o) \in X \times X$. Therefore, it must contain a rectangular neighbourhood $W \times W'$ where $W, W' \in \mathcal{F}$. Taking $V = W \cap W'$ we get the conclusion, i.e. $V + V \subset U$.
3. By Proposition 2.1.7, fixed an arbitrary $0 \neq \lambda \in \mathbb{K}$, the map $x \mapsto \lambda^{-1}x$ of X into itself is continuous. Therefore, the preimage of any neighbourhood U of the origin must be also such a neighbourhood. This preimage is clearly λU, hence $\lambda U \in \mathcal{F}$.
4. Suppose by contradiction that U is not absorbing. Then there exists $y \in X$ such that $\forall n \in \mathbb{N}$ we have that $\frac{1}{n}y \notin U$. This contradicts the convergence of $\frac{1}{n}y \to o$ as $n \to \infty$ (because $U \in \mathcal{F}$ must contain infinitely many terms of the sequence $(\frac{1}{n}y)_{n\in\mathbb{N}}$.
5. Since by the definition of t.v.s. the scalar multiplication $\mathbb{K} \times X \to X$, $(\lambda, x) \mapsto \lambda x$ is continuous, the preimage of U under this map must be a neighbourhood of $(0, o) \in \mathbb{K} \times X$. Therefore, it contains a rectangular neighbourhood $N \times W$ where N is a neighbourhood of 0 in the euclidean topology on $\mathbb{K}$ and $W \in \mathcal{F}$. On the other hand, there exists $\rho > 0$ such that $B_\rho(0) := \{\lambda \in \mathbb{K} : |\lambda| \leq \rho\} \subseteq N$. Thus $B_\rho(0) \times W$ is contained in the preimage of U under the scalar multiplication, i.e. $\lambda W \subset U$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$. Hence, the set $V = \cup_{|\lambda| \leq \rho} \lambda W \subset U$. Now $V \in \mathcal{F}$ since each $\lambda W \in \mathcal{F}$ by 3 and V is clearly balanced (since for any $x \in V$ there exists $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$ s.t. $x \in \lambda W$ and therefore for any $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$ we get $\alpha x \in \alpha\lambda W \subset V$ because $|\alpha\lambda| \leq \rho$).

<u>*Sufficiency part.*</u>
Suppose that the conditions 1,2,3,4,5 hold for a filter $\mathcal{F}$ of the vector space X. We want to show that there exists a topology τ on X such that $\mathcal{F}$ is the filter of neighbourhoods of the origin w.r.t. to τ and (X, τ) is a t.v.s. according to Definition 2.1.1.

Let us define for any $x \in X$ the filter $\mathcal{F}(x) := \{U + x : U \in \mathcal{F}\}$. It is easy to see that $\mathcal{F}(x)$ fulfills the properties (N1) and (N2) of Theorem 1.1.9. In fact, we have:

- By 1 we have that $\forall U \in \mathcal{F}$, $o \in U$, then $\forall U \in \mathcal{F}$, $x = o + x \in U + x$, i.e. $\forall A \in \mathcal{F}(x)$, $x \in A$.
- Let $A \in \mathcal{F}(x)$ then $A = U + x$ for some $U \in \mathcal{F}$. By 2, we have that there exists $V \in \mathcal{F}$ s.t. $V + V \subset U$. Define $B := V + x \in \mathcal{F}(x)$ and take any $y \in B$ then we have $V + y \subset V + B \subset V + V + x \subset U + x = A$. But $V + y$ belongs to the filter $\mathcal{F}(y)$ and therefore so does A.

By Theorem 1.1.9, there exists a unique topology τ on X such that $\mathcal{F}(x)$ is the filter of neighbourhoods of each point $x \in X$ and so for which in particular $\mathcal{F}$ is the filter of neighbourhoods of the origin.

It remains to prove that the vector addition and the scalar multiplication in X are continuous w.r.t. to τ.

- The continuity of the addition easily follows from the property 2. Indeed, let $(x_0, y_0) \in X \times X$ and take a neighbourhood W of its image $x_0 + y_0$. Then $W = U + x_0 + y_0$ for some $U \in \mathcal{F}$. By 2, there exists $V \in \mathcal{F}$ s.t. $V + V \subset U$ and so $(V + x_0) + (V + y_0) \subset W$. This implies that the preimage of W under the addition contains $(V + x_0) \times (V + y_0)$ which is a neighbourhood of (x_0, y_0).
- To prove the continuity of the scalar multiplication, let $(\lambda_0, x_0) \in \mathbb{K} \times X$ and take a neighbourhood U' of $\lambda_0 x_0$. Then $U' = U + \lambda_0 x_0$ for some $U \in \mathcal{F}$. By 2 and 5, there exists $W \in \mathcal{F}$ s.t. $W + W + W \subset U$ and W is balanced. By 4, W is also absorbing so there exists $\rho > 0$ (w.l.o.g we can take $\rho \leq 1$) such that $\forall \lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$ we have $\lambda x_0 \in W$.

  Suppose $\lambda_0 = 0$ then $\lambda_0 x_0 = o$ and $U' = U$. Now
  $$Im(B_\rho(0) \times (W + x_0)) = \{\lambda y + \lambda x_0 : \lambda \in B_\rho(0), y \in W\}.$$
  As $\lambda \in B_\rho(0)$ and W is absorbing, $\lambda x_0 \in W$. Also since $|\lambda| \leq \rho \leq 1$ for all $\lambda \in B_\rho(0)$ and since W is balanced, we have $\lambda W \subset W$. Thus $Im(B_\rho(0) \times (W + x_0)) \subset W + W \subset W + W + W \subset U$ and so the preimage of U under the scalar multiplication contains $B_\rho(0) \times (W + x_0)$ which is a neighbourhood of $(0, x_0)$.

  Suppose $\lambda_0 \neq 0$ and take $\sigma = \min\{\rho, |\lambda_0|\}$. Then $Im((B_\sigma(0) + \lambda_0) \times (|\lambda_0|^{-1} W + x_0)) = \{\lambda |\lambda_0|^{-1} y + \lambda x_0 + \lambda_0 |\lambda_0|^{-1} y + \lambda x_0 : \lambda \in B_\sigma(0), y \in W\}$. As $\lambda \in B_\sigma(0)$, $\sigma \leq \rho$ and W is absorbing, $\lambda x_0 \in W$. Also since $\forall \lambda \in B_\sigma(0)$ the modulus of $\lambda |\lambda_0|^{-1}$ and $\lambda_0 |\lambda_0|^{-1}$ are both ≤ 1 and since W is balanced, we have $\lambda |\lambda_0|^{-1} W, \lambda_0 |\lambda_0|^{-1} W \subset W$. Thus $Im(B_\sigma(0) + \lambda_0 \times (|\lambda_0|^{-1} W + x_0)) \subset W + W + W + \lambda_0 x_0 \subset U + \lambda_0 x_0$ and so the preimage of $U + \lambda_0 x_0$ under the scalar multiplication contains $B_\sigma(0) + \lambda_0 \times (|\lambda_0|^{-1} W + x_0)$ which is a neighbourhood of (λ_0, x_0). □

It easily follows from previous theorem that:

Corollary 2.1.14.

a) Every t.v.s. has always a base of closed neighbourhoods of the origin.
b) Every t.v.s. has always a base of balanced absorbing neighbourhoods of the origin. In particular, it has always a base of closed balanced absorbing neighbourhoods of the origin.
c) Proper subspaces of a t.v.s. are never absorbing. In particular, if M is an open subspace of a t.v.s. X then $M = X$.

Proof. (Sheet 3, Exercise 3) □

Let us show some further useful properties of the t.v.s.:

Proposition 2.1.15.

1. *Every linear subspace of a t.v.s. endowed with the correspondent subspace topology is itself a t.v.s..*
2. *The closure $\overline{H}$ of a linear subspace H of a t.v.s. X is again a linear subspace of X.*
3. *Let X, Y be two t.v.s. and $f : X \to Y$ a linear map. f is continuous if and only if f is continuous at the origin o.*

Proof.

1. This clearly follows by the fact that the addition and the multiplication restricted to the subspace are just a composition of continuous maps (recall that inclusion is continuous in the subspace topology c.f. Definition 1.1.17).
2. Let $x_0, y_0 \in \overline{H}$ and let us take any $U \in \mathcal{F}(o)$. By Theorem 2.1.10-2, there exists $V \in \mathcal{F}(o)$ s.t. $V + V \subset U$. Then, by definition of closure points, there exist $x, y \in H$ s.t. $x \in V + x_0$ and $y \in V + y_0$. Therefore, we have that $x + y \in H$ (since H is a linear subspace) and $x+y \in (V+x_0)+(V+y_0) \subset U+x_0+y_0$. Hence, $x_0+y_0 \in \overline{H}$. Similarly, one can prove that if $x_0 \in \overline{H}$, $\lambda x_0 \in \bar{H}$ for any $\lambda \in \mathbb{K}$.
3. Assume that f is continuous at $o \in X$ and fix any $x \neq o$ in X. Let U be an arbitrary neighbourhood of $f(x) \in Y$. By Corollary 2.1.9, we know that $U = f(x) + V$ where V is a neighbourhood of $o \in Y$. Since f is linear we have that:

   $$f^{-1}(U) = f^{-1}(f(x) + V) \supset x + f^{-1}(V).$$

   By the continuity at the origin of X, we know that $f^{-1}(V)$ is a neighbourhood of $o \in X$ and so $x + f^{-1}(V)$ is a neighbourhood of $x \in X$.

□

2.2 Hausdorff topological vector spaces

For convenience let us recall here the definition of Hausdorff space.

Definition 2.2.1. *A topological space X is said to be* Hausdorff *or* (T2) *if any two distinct points of X have neighbourhoods without common points; or equivalently if two distinct points always lie in disjoint open sets.*

Note that in a Hausdorff space, any set consisting of a single point is closed but there are topological spaces with the same property which are not Hausdorff and we will see in this section that such spaces are not t.v.s..

Definition 2.2.2. *A topological space X is said to be* (T1) *if, given two distinct points of X, each lies in a neighborhood which does not contain the other point; or equivalently if, for any two distinct points, each of them lies in an open subset which does not contain the other point.*

It is easy to see that in a topological space which is (T1) all singletons are closed (Sheet 4, Exercise 2).

From the definition it is clear that (T2) implies (T1) but in general the inverse does not hold (c.f. Examples 1.1.40-4 for an example of topological space which is T1 but not T2). However, the following results shows that a t.v.s is Hausdorff if and only if it is (T1).

Proposition 2.2.3. *A t.v.s. X is Hausdorff iff*

$$\forall\, o \neq x \in X,\ \exists\, U \in \mathcal{F}(o)\ s.t.\ x \notin U. \tag{2.1}$$

Proof.

($\Rightarrow$) Let (X, τ) be Hausdorff. Then there exist $U \in \mathcal{F}(o)$ and $V \in \mathcal{F}(x)$ s.t. $U \cap V = \emptyset$. This means in particular that $x \notin U$.

($\Leftarrow$) Assume that (2.1) holds and let $x, y \in X$ with $x \neq y$, i.e. $x - y \neq o$. Then there exists $U \in \mathcal{F}(o)$ s.t. $x - y \notin U$. By (2) and (5) of Theorem 2.1.10, there exists $V \in \mathcal{F}(o)$ balanced and s.t. $V + V \subset U$. Since V is balanced $V = -V$ then we have $V - V \subset U$. Suppose now that $(V + x) \cap (V + y) \neq \emptyset$, then there exists $z \in (V + x) \cap (V + y)$, i.e. $z = v + x = w + y$ for some $v, w \in V$. Then $x - y = w - v \in V - V \subset U$ and so $x - y \in U$ which is a contradiction. Hence, $(V + x) \cap (V + y) = \emptyset$ and by Corollary 2.1.9 we know that $V + x \in \mathcal{F}(x)$ and $V + y \in \mathcal{F}(y)$. Hence, X is (T2). □

Note that since the topology of a t.v.s. is translation invariant then the previous proposition guarantees that *a t.v.s is Hausdorff iff it is (T1).* As a matter of fact, we have the following result:

Corollary 2.2.4. *For a t.v.s. X the following are equivalent:*
a) X is Hausdorff.
b) the intersection of all neighbourhoods of the origin o is just $\{o\}$.
c) $\{0\}$ is closed.

Note that in a t.v.s. $\{0\}$ is closed is equivalent to say that all singletons are closed (and so that the space is (T1)).

Proof.
a)$\Rightarrow$ b) Let X be a Hausdorff t.v.s. space. Clearly, $\{o\} \subseteq \cap_{U\in\mathcal{F}(o)}U$. Now if b) does not hold, then there exists $x \in \cap_{U\in\mathcal{F}(o)}U$ with $x \neq o$. But by the previous theorem we know that (2.1) holds and so there exists $V \in \mathcal{F}(o)$ s.t. $x \notin V$ and so $x \notin \cap_{U\in\mathcal{F}(o)}U$ which is a contradiction.

b)$\Rightarrow$ c) Assume that $\cap_{U\in\mathcal{F}(o)}U = \{o\}$. If $x \in \overline{\{o\}}$ then $\forall\, V_x \in \mathcal{F}(x)$ we have $V_x \cap \{o\} \neq \emptyset$, i.e. $o \in V_x$. By Corollary 2.1.9 we know that each $V_x = U + x$ with $U \in \mathcal{F}(o)$. Then $o = u + x$ for some $u \in U$ and so $x = -u \in -U$. This means that $x \in \cap_{U\in\mathcal{F}(o)}(-U)$. Since every dilation is an homeomorphism and b) holds, we have that $x \in \cap_{U\in\mathcal{F}(o)}U = \{0\}$. Hence, $x = 0$ and so $\overline{\{o\}} = \{o\}$, i.e. $\{o\}$ is closed.

c)$\Rightarrow$ a) Assume that X is not Hausdorff. Then by the previous proposition (2.1) does not hold, i.e. there exists $x \neq o$ s.t. $x \in U$ for all $U \in \mathcal{F}(o)$. This means that $x \in \cap_{U\in\mathcal{F}(o)}U \subseteq \cap_{U\in\mathcal{F}(o)\text{closed}}U = \overline{\{o\}}$ By c), $\overline{\{o\}} = \{o\}$ and so $x = 0$ which is a contradiction. ☐

Example 2.2.5. *Every vector space with an infinite number of elements endowed with the cofinite topology is not a tvs.* It is clear that in such topological space all singletons are closed (i.e. it is T1). Therefore, if it was a t.v.s. then by the previous results it should be a Hausdorff space which is not true as shown in Example 1.1.40.

2.3 Quotient topological vector spaces

Quotient topology

Let X be a topological space and $\sim$ be any equivalence relation on X. Then the *quotient set* $X/\!\sim$ is defined to be the set of all equivalence classes w.r.t. to $\sim$. The map $\phi : X \to X/\!\sim$ which assigns to each $x \in X$ its equivalence class $\phi(x)$ w.r.t. $\sim$ is called the *canonical map* or *quotient map*. Note that ϕ is surjective. We may define a topology on $X/\sim$ by setting that: a subset U of $X/\!\sim$ is open iff the preimage $\phi^{-1}(U)$ is open in X. This is called the *quotient topology* on $X/\sim$. Then it is easy to verify (Sheet 4, Exercise 2) that:

- the quotient map ϕ is continuous.
- the quotient topology on $X/\sim$ is the finest topology on $X/\sim$ s.t. ϕ is continuous.