Corollary 2.2.4. For a t.v.s. $X$ the following are equivalent:

a) $X$ is Hausdorff.

b) the intersection of all neighbourhoods of the origin $o$ is just $\{o\}$.

c) $\{0\}$ is closed.

Note that in a t.v.s. $\{0\}$ is closed is equivalent to say that all singletons are closed (and so that the space is (T1)).

Proof. a) $\Rightarrow$ b) Let $X$ be a Hausdorff t.v.s. space. Clearly, $\{o\} \subseteq \cap_{U \in F(o)} U$. Now if b) does not hold, then there exists $x \in \cap_{U \in F(o)} U$ with $x \neq o$. But by the previous theorem we know that (2.1) holds and so there exists $V \in F(o)$ s.t. $x \notin V$ and so $x \notin \cap_{U \in F(o)} U$ which is a contradiction.

b) $\Rightarrow$ c) Assume that $\cap_{U \in F(o)} U = \{o\}$. If $x \in \overline{\{o\}}$ then $\forall V_x \in F(x)$ we have $V_x \cap \{o\} \neq \emptyset$, i.e. $o \in V_x$. By Corollary 2.1.9 we know that each $V_x = U + x$ with $U \in F(o)$. Then $o = u + x$ for some $u \in U$ and so $x = -u \in -U$. This means that $x \in \cap_{U \in F(o)} (-U)$. Since every dilation is an homeomorphism and b) holds, we have that $x \in \cap_{U \in F(o)} U = \{0\}$. Hence, $x = 0$ and so $\overline{\{o\}} = \{o\}$, i.e. $\{o\}$ is closed.

c) $\Rightarrow$ a) Assume that $X$ is not Hausdorff. Then by the previous proposition (2.1) does not hold, i.e. there exists $x \neq o$ s.t. $x \in U$ for all $U \in F(o)$. This means that $x \in \cap_{U \in F(o)} U \subseteq \cap_{U \in F(o)} U = \{o\}$ By c), $\overline{\{o\}} = \{o\}$ and so $x = 0$ which is a contradiction.

Example 2.2.5. Every vector space with an infinite number of elements endowed with the cofinite topology is not a t.v.s. It is clear that in such topological space all singletons are closed (i.e. it is T1). Therefore, if it was a t.v.s. then by the previous results it should be a Hausdorff space which is not true as shown in Example 1.1.40.

2.3 Quotient topological vector spaces

Quotient topology

Let $X$ be a topological space and $\sim$ be any equivalence relation on $X$. Then the quotient set $X/\sim$ is defined to be the set of all equivalence classes w.r.t. to $\sim$. The map $\phi : X \to X/\sim$ which assigns to each $x \in X$ its equivalence class $\phi(x)$ w.r.t. $\sim$ is called the canonical map or quotient map. Note that $\phi$ is surjective. We may define a topology on $X/\sim$ by setting that: a subset $U$ of $X/\sim$ is open iff the preimage $\phi^{-1}(U)$ is open in $X$. This is called the quotient topology on $X/\sim$. Then it is easy to verify (Sheet 4, Exercise 2) that:
2. **Topological Vector Spaces**

- the quotient map \( \phi \) is continuous.
- the quotient topology on \( X/\sim \) is the finest topology on \( X/\sim \) s.t. \( \phi \) is continuous.

Note that the quotient map \( \phi \) is not necessarily open or closed.

**Example 2.3.1.** Consider \( \mathbb{R} \) with the standard topology given by the modulus and define the following equivalence relation on \( \mathbb{R} \):

\[ x \sim y \iff (x = y \lor \{x, y\} \subset \mathbb{Z}). \]

Let \( \mathbb{R}/\sim \) be the quotient set w.r.t \( \sim \) and \( \phi : \mathbb{R} \to \mathbb{R}/\sim \) the correspondent quotient map. Let us consider the quotient topology on \( \mathbb{R}/\sim \). Then \( \phi \) is not an open map. In fact, if \( U \) is an open proper subset of \( \mathbb{R} \) containing an integer, then \( \phi^{-1}(\phi(U)) = U \cup \mathbb{Z} \) which is not open in \( \mathbb{R} \) with the standard topology. Hence, \( \phi(U) \) is not open in \( \mathbb{R}/\sim \) with the quotient topology.

For an example of quotient map which is not closed see Example 2.3.3 in the following.

**Quotient vector space**

Let \( X \) be a vector space and \( M \) a linear subspace of \( X \). For two arbitrary elements \( x, y \in X \), we define \( x \sim_M y \) iff \( x - y \in M \). It is easy to see that \( \sim_M \) is an equivalence relation: it is reflexive, since \( x - x = 0 \in M \) (every linear subspace contains the origin); it is symmetric, since \( x - y \in M \) implies \( -(x - y) = y - x \in M \) (if a linear subspace contains an element, it contains its inverse); it is transitive, since \( x - y \in M \), \( y - z \in M \) implies \( x - z = (x - y) + (y - z) \in M \) (when a linear subspace contains two vectors, it also contains their sum). Then \( X/M \) is defined to be the quotient set \( X/\sim_M \), i.e. the set of all equivalence classes for the relation \( \sim_M \) described above. The canonical (or quotient) map \( \phi : X \to X/M \) which assigns to each \( x \in X \) its equivalence class \( \phi(x) \) w.r.t. the relation \( \sim_M \) is clearly surjective. Using the fact that \( M \) is a linear subspace of \( X \), it is easy to check that:

1. if \( x \sim_M y \), then \( \forall \lambda \in \mathbb{K} \) we have \( \lambda x \sim_M \lambda y \).
2. if \( x \sim_M y \), then \( \forall z \in X \) we have \( x + z \sim_M y + z \).

These two properties guarantee that the following operations are well-defined on \( X/M \):

- vector addition: \( \forall \phi(x), \phi(y) \in X/M, \phi(x) + \phi(y) := \phi(x + y) \)
- scalar multiplication: \( \forall \lambda \in \mathbb{K}, \forall \phi(x) \in X/M, \lambda \phi(x) := \phi(\lambda x) \)

\( X/M \) with the two operations defined above is a vector space and therefore it is often called *quotient vector space*. Then the quotient map \( \phi \) is clearly linear.
2.3. Quotient topological vector spaces

**Quotient topological vector space**

Let $X$ be now a t.v.s. and $M$ a linear subspace of $X$. Consider the quotient vector space $X/M$ and the quotient map $\phi : X \to X/M$ defined in Section 2.3. Since $X$ is a t.v.s, it is in particular a topological space, so we can consider on $X/M$ the quotient topology defined in Section 2.3. We already know that in this topological setting $\phi$ is continuous but actually the structure of t.v.s. on $X$ guarantees also that it is open.

**Proposition 2.3.2.** For a linear subspace $M$ of a t.v.s $X$, the quotient mapping $\phi : X \to X/M$ is open (i.e. carries open sets in $X$ to open sets in $X/M$) when $X/M$ is endowed with the quotient topology.

**Proof.** Let $V$ open in $X$. Then we have

$$\phi^{-1}(\phi(V)) = V + M = \bigcup_{m \in M} (V + m)$$

Since $X$ is a t.v.s, its topology is translation invariant and so $V + m$ is open for any $m \in M$. Hence, $\phi^{-1}(\phi(V))$ is open in $X$ as union of open sets. By definition, this means that $\phi(V)$ is open in $X/M$ endowed with the quotient topology. 

It is then clear that $\phi$ carries neighborhoods of a point in $X$ into neighborhoods of a point in $X/M$ and viceversa. Hence, the neighborhoods of the origin in $X/M$ are direct images under $\phi$ of the neighborhoods of the origin in $X$. In conclusion, when $X$ is a t.v.s and $M$ is a subspace of $X$, we can rewrite the definition of quotient topology on $X/M$ in terms of neighborhoods as follows: the filter of neighborhoods of the origin of $X/M$ is exactly the image under $\phi$ of the filter of neighborhoods of the origin in $X$.

It is not true, in general (not even when $X$ is a t.v.s. and $M$ is a subspace of $X$), that the quotient map is closed.

**Example 2.3.3.**

Consider $\mathbb{R}^2$ with the euclidean topology and the hyperbola $H := \{(x, y) \in \mathbb{R}^2 : xy = 1\}$. If $M$ is one of the coordinate axes, then $\mathbb{R}^2/M$ can be identified with the other coordinate axis and the quotient map $\phi$ with the orthogonal projection on it. All these identifications are also valid for the topologies. The hyperbola $H$ is closed in $\mathbb{R}^2$ but its image under $\phi$ is the complement of the origin on a straight line which is open.
Corollary 2.3.4. For a linear subspace \( M \) of a t.v.s. \( X \), the quotient space \( X/M \) endowed with the quotient topology is a t.v.s.

Proof.  
For convenience, we denote here by \( A \) the vector addition in \( X/M \) and just by + the vector addition in \( X \). Let \( W \) be a neighbourhood of the origin \( o \) in \( X/M \). We aim to prove that \( A^{-1}(W) \) is a neighbourhood of \((o, o)\) in \( X/M \times X/M \).

The continuity of the quotient map \( \phi : X \to X/M \) implies that \( \phi^{-1}(W) \) is a neighbourhood of the origin in \( X \). Then, by Theorem 2.1.10-2 (we can apply the theorem because \( X \) is a t.v.s.), there exists \( V \) neighbourhood of the origin in \( X \) s.t. \( V + V \subseteq \phi^{-1}(W) \). Hence, by the linearity of \( \phi \), we get \( A(\phi(V) \times \phi(V)) = \phi(V + V) \subseteq W \), i.e. \( \phi(V) \times \phi(V) \subseteq A^{-1}(W) \). Since \( \phi \) is also open, \( \phi(V) \) is a neighbourhood of the origin \( o \) in \( X/M \) and so \( A^{-1}(W) \) is a neighbourhood of \((o, o)\) in \( X/M \times X/M \).

A similar argument gives the continuity of the scalar multiplication.

Proposition 2.3.5. Let \( X \) be a t.v.s. and \( M \) a linear subspace of \( X \). Consider \( X/M \) endowed with the quotient topology. Then the two following properties are equivalent:

a) \( M \) is closed  
b) \( X/M \) is Hausdorff

Proof.  
In view of Corollary 2.2.4, (b) is equivalent to say that the complement of the origin in \( X/M \) is open w.r.t. the quotient topology. But the complement of the origin in \( X/M \) is exactly the image under \( \phi \) of the complement of \( M \) in \( X \). Since \( \phi \) is an open continuous map, the image under \( \phi \) of the complement of \( M \) in \( X \) is open in \( X/M \) iff the complement of \( M \) in \( X \) is open, i.e. (a) holds.

Corollary 2.3.6. If \( X \) is a t.v.s., then \( X/\{o\} \) endowed with the quotient topology is a Hausdorff t.v.s.. \( X/\{o\} \) is said to be the Hausdorff t.v.s. associated with the t.v.s. \( X \). When a t.v.s. \( X \) is Hausdorff, \( X \) and \( X/\{o\} \) are topologically isomorphic.

Proof.  
Since \( X \) is a t.v.s. and \( \{o\} \) is a linear subspace of \( X \), \( \overline{\{o\}} \) is a closed linear subspace of \( X \). Then, by Corollary 2.3.4 and Proposition 2.3.5, \( X/\{o\} \) is a Hausdorff t.v.s.. If in addition \( X \) is Hausdorff, then Corollary 2.2.4 guarantees that \( \overline{\{o\}} = \{o\} \) in \( X \). Therefore, the quotient map \( \phi : X \to X/\{o\} \) is also injective because in this case \( \text{Ker}(\phi) = \{o\} \). Hence, \( \phi \) is a topological isomorphism (i.e. bijective, continuous, open, linear) between \( X \) and \( X/\{o\} \) which is indeed \( X/\{o\} \).

\[ \square \]
2.4 Continuous linear mappings between t.v.s.

Let $X$ and $Y$ be two vector spaces over $\mathbb{K}$ and $f : X \to Y$ a linear map. We define the image of $f$, and denote it by $\text{Im}(f)$, as the subset of $Y$:

$$\text{Im}(f) := \{ y \in Y : \exists x \in X \text{ s.t. } y = f(x) \}.$$ 

We define the kernel of $f$, and denote it by $\text{Ker}(f)$, as the subset of $X$:

$$\text{Ker}(f) := \{ x \in X : f(x) = 0 \}.$$ 

Both $\text{Im}(f)$ and $\text{Ker}(f)$ are linear subspaces of $Y$ and $X$, respectively. We have then the diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & \text{Im}(f) & \xrightarrow{i} & Y \\
\downarrow{\phi} & & & & \downarrow{f} \\
X/\text{Ker}(f) & & & & \\
\end{array}
$$

where $i$ is the natural injection of $\text{Im}(f)$ into $Y$, i.e. the mapping which to each element $y$ of $\text{Im}(f)$ assigns that same element $y$ regarded as an element of $Y$; $\phi$ is the canonical map of $X$ onto its quotient $X/\text{Ker}(f)$. The mapping $\tilde{f}$ is defined so as to make the diagram commutative, which means that:

$$\forall x \in X, f(x) = \tilde{f}(\phi(x)).$$

Note that

- $\tilde{f}$ is well-defined.
  Indeed, if $\phi(x) = \phi(y)$, i.e. $x - y \in \text{Ker}(f)$, then $f(x - y) = 0$ that is $f(x) = f(y)$ and so $\tilde{f}(\phi(x)) = \tilde{f}(\phi(y))$.

- $\tilde{f}$ is linear.
  This is an immediate consequence of the linearity of $f$ and of the linear structure of $X/\text{Ker}(f)$.

- $\tilde{f}$ is a one-to-one linear map of $X/\text{Ker}(f)$ onto $\text{Im}(f)$.
  The onto property is evident from the definition of $\text{Im}(f)$ and of $\tilde{f}$.
  As for the one-to-one property, note that $\tilde{f}(\phi(x)) = \tilde{f}(\phi(y))$ means by definition that $f(x) = f(y)$, i.e. $f(x - y) = 0$. This is equivalent, by linearity of $f$, to say that $x - y \in \text{Ker}(f)$, which means that $\phi(x) = \phi(y)$.
2. Topological Vector Spaces

The set of all linear maps (continuous or not) of a vector space $X$ into another vector space $Y$ is denoted by $\mathcal{L}(X; Y)$. Note that $\mathcal{L}(X; Y)$ is a vector space for the natural addition and multiplication by scalars of functions. Recall that when $Y = \mathbb{K}$, the space $\mathcal{L}(X; Y)$ is denoted by $X^*$ and it is called the algebraic dual of $X$ (see Definition 1.2.4).

Let us not turn to consider linear mapping between two t.v.s. $X$ and $Y$. Since they possess a topological structure, it is natural to study in this setting continuous linear mapping.

**Proposition 2.4.1.** Let $f : X \to Y$ a linear map between two t.v.s. $X$ and $Y$. If $Y$ is Hausdorff and $f$ is continuous, then $\ker(f)$ is closed in $X$.

**Proof.**
Clearly, $\ker(f) = f^{-1}(\{0\})$. Since $Y$ is a Hausdorff t.v.s., $\{0\}$ is closed in $Y$ and so, by the continuity of $f$, $\ker(f)$ is also closed in $Y$. $\square$

Note that $\ker(f)$ might be closed in $X$ also when $Y$ is not Hausdorff. For instance, when $f \equiv 0$ or when $f$ is injective and $X$ is Hausdorff.

**Proposition 2.4.2.** Let $f : X \to Y$ a linear map between two t.v.s. $X$ and $Y$. The map $f$ is continuous if and only if the map $\bar{f}$ is continuous.

**Proof.**
Suppose $f$ continuous and let $U$ be an open subset in $\text{Im}(f)$. Then $f^{-1}(U)$ is open in $X$. By definition of $\bar{f}$, we have $f^{-1}(U) = \phi(f^{-1}(U))$. Since the quotient map $\phi : X \to X/\ker(f)$ is open, $\phi(f^{-1}(U))$ is open in $X/\ker(f)$. Hence, $f^{-1}(U)$ is open in $X/\ker(f)$ and so the map $\bar{f}$ is continuous. Viceversa, suppose that $\bar{f}$ is continuous. Since $f = \bar{f} \circ \phi$ and $\phi$ is continuous, $f$ is also continuous as composition of continuous maps. $\square$

In general, the inverse of $\bar{f}$, which is well defined on $\text{Im}(f)$ since $\bar{f}$ is injective, is not continuous. In other words, $\bar{f}$ is not necessarily bi-continuous.

The set of all continuous linear maps of a t.v.s. $X$ into another t.v.s. $Y$ is denoted by $L(X; Y)$ and it is a vector subspace of $\mathcal{L}(X; Y)$. When $Y = \mathbb{K}$, the space $L(X; Y)$ is usually denoted by $X'$ which is called the topological dual of $X$, in order to underline the difference with $X^*$ the algebraic dual of $X$. $X'$ is a vector subspace of $X^*$ and is exactly the vector space of all continuous linear functionals, or continuous linear forms, on $X$. The vector spaces $X'$ and $L(X; Y)$ will play an important role in the forthcoming and will be equipped with various topologies.