Proof. of Proposition 2.5.8

a) Let A be a complete subset of a Hausdorff t.v.s. X and let $x \in \overline{A}$. By Lemma 2.5.11, $x \in \overline{A}$ implies that there exists a filter $\mathcal{F}$ of subsets of X s.t. $A \in \mathcal{F}$ and $\mathcal{F}$ converges to x. Therefore, by Proposition 2.5.4-c), $\mathcal{F}$ is a Cauchy filter. Consider now $\mathcal{F}_A := \{U \in \mathcal{F} \,:\, U \subseteq A\} \subset \mathcal{F}$. It is easy to see that $\mathcal{F}_A$ is a Cauchy filter on A and so the completeness of A ensures that $\mathcal{F}_A$ converges to a point $y \in A$. Hence, any nbhood V of y in A belongs to $\mathcal{F}_A$ and so to $\mathcal{F}$. By definition of subset topology, this means that for any nbhood U of y in X we have $U \cap A \in \mathcal{F}$ and so $U \in \mathcal{F}$ (since $\mathcal{F}$ is a filter). Then $\mathcal{F}$ converges to y. Since X is Hausdorff, Lemma 2.5.10 establishes the uniqueness of the limit point of $\mathcal{F}$, i.e. $x = y$ and so $\overline{A} = A$.

b) Let A be a closed subset of a complete t.v.s. X and let $\mathcal{F}_A$ be any Cauchy filter on A. Take the filter $\mathcal{F} := \{F \subseteq X | B \subseteq F \text{ for some } B \in \mathcal{F}_A\}$. It is clear that $\mathcal{F}$ contains A and is finer than the Cauchy filter $\mathcal{F}_A$. Therefore, by Proposition 2.5.4-b), $\mathcal{F}$ is also a Cauchy filter. Then the completeness of the t.v.s. X gives that $\mathcal{F}$ converges to a point $x \in X$, i.e. $\mathcal{F}(x) \subseteq \mathcal{F}$. By Lemma 2.5.11, this implies that actually $x \in \overline{A}$ and, since A is closed, that $x \in A$. Now any neighbourhood of $x \in A$ in the subset topology is of the form $U \cap A$ with $U \in \mathcal{F}(x)$. Since $\mathcal{F}(x) \subseteq \mathcal{F}$ and $A \in \mathcal{F}$, we have $U \cap A \in \mathcal{F}$. Therefore, there exists $B \in \mathcal{F}_A$ s.t. $B \subseteq U \cap A \subset A$ and so $U \cap A \in \mathcal{F}_A$. Hence, $\mathcal{F}_A$ converges $x \in A$, i.e. A is complete. □

When a t.v.s. is not complete, it makes sense to ask if it is possible to embed it in a complete one. We are going to describe an abstract procedure that allows to always associate to an arbitrary Hausdorff t.v.s. X a complete Hausdorff t.v.s. $\hat{X}$ called the *completion* of X. Before doing that, we need to introduce uniformly continuous functions between t.v.s. and state some of their fundamental properties.

Definition 2.5.12. *Let X and Y be two t.v.s. and let A be a subset of X. A mapping $f : A \to Y$ is said to be* uniformly continuous *if for every neighborhood V of the origin in Y, there exists a neighborhood U of the origin in X such that for all pairs of elements $x_1, x_2 \in A$*

$$x_1 - x_2 \in U \;\Rightarrow\; f(x_1) - f(x_2) \in V.$$

Proposition 2.5.13. *Let X and Y be two t.v.s. and let A be a subset of X.*

a) If $f : A \to Y$ is uniformly continuous, then the image under f of a Cauchy filter on A is a Cauchy filter on Y.

b) If A is a linear subspace of X, then every continuous linear map from A to Y is uniformly continuous.

Proof. (Sheet 6, Exercise 2) □

Theorem 2.5.14.
Let X and Y be two Hausdorff t.v.s., A a dense subset of X, and $f : A \to Y$ a uniformly continuous mapping. If Y is complete the the following hold.

a) There exists a unique continuous mapping $\bar{f} : X \to Y$ which extends f, i.e. such that for all $x \in A$ we have $\bar{f}(x) = f(x)$.

b) $\bar{f}$ is uniformly continuous.

c) If we additionally assume that f is linear and A is a linear subspace of X, then $\bar{f}$ is linear.

Proof. (Sheet 6, Exercise 3) □

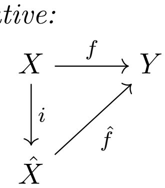
Let us now state and prove the theorem on completion of a t.v.s..

Theorem 2.5.15.
Let X be a Haudorff t.v.s.. Then there exists a complete Hausdorff t.v.s. $\hat{X}$ and a mapping $i : X \to \hat{X}$ with the following properties:

a) The mapping i is a topological monomorphism.

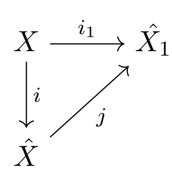
b) The image of X under i is dense in $\hat{X}$.

c) For every complete Hausdorff t.v.s. Y and for every continuous linear map $f : X \to Y$, there is a continuous linear map $\hat{f} : \hat{X} \to Y$ such that the following diagram is commutative:



Furthermore:

I) Any other pair $(\hat{X}_1, i_1)$, consisting of a complete Hausdorff t.v.s. $\hat{X}_1$ and of a mapping $i_1 : X \to \hat{X}_1$ such that properties (a) and (b) hold substituting $\hat{X}$ with $\hat{X}_1$ and i with i_1, is isomorphic to $(\hat{X}, i)$. This means that there is an isomorphism j of $\hat{X}$ onto $\hat{X}_1$ such that the following diagram is commutative:



II) Given Y and f as in property (c), the continuous linear map $\hat{f}$ is unique.

Proof.
1) The set $\hat{X}$
Define the following relation on the collection of all Cauchy filters (c.f.) on X:

$$\mathcal{F} \sim_R \mathcal{G} \Leftrightarrow \forall\, U \text{ nbhood of the origin in } X,\ \exists\, A \in \mathcal{F},\ \exists\, B \in \mathcal{G} \text{ s.t. } A - B \subset U.$$

The relation (R) is actually an equivalence relation. In fact:

- reflexive: If $\mathcal{F}$ is a c.f. on X, then by Definition 2.5.2 we have that for any U nbhood of the origin in X there exists $A \in \mathcal{F}$ s.t. $A - A \subset U$, i.e. $\mathcal{F} \sim_R \mathcal{F}$.
- symmetric: If $\mathcal{F}$ and $\mathcal{G}$ are c.f. on X s.t. $\mathcal{F} \sim_R \mathcal{G}$, then by definition of (R) we have that for any U nbhood of the origin in X there exist $A \in \mathcal{F}$ and $B \in \mathcal{G}$ s.t. $A - B \subset U$. This implies that $B - A \subset -U$, which gives $\mathcal{G} \sim_R \mathcal{F}$ considering that $-U$ is a generic nbhood of the origin in X int he same right as U.
- transitive: Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be c.f. on X s.t. $\mathcal{F} \sim_R \mathcal{G}$ and $\mathcal{G} \sim_R \mathcal{H}$. Take any U nbhood of the origin in X, then Theorem 2.1.10 ensures that there exists V nbhood of the origin in X s.t. $V + V \subset U$. By definition of (R), there exists $A \in \mathcal{F}$, $B_1, B_2 \in \mathcal{G}$ and $C \in \mathcal{H}$ s.t. $A - B_1 \subset V$ and $B_2 - C \subset V$. This clearly implies $A - (B_1 \cap B_2) \subset V$ and $(B_1 \cap B_2) - C \subset V$. By adding we obtain
$$A - C \subset A - (B_1 \cap B_2) + (B_1 \cap B_2) - C \subset V + V \subset U.$$

We define $\hat{X}$ as the quotient of the set of all c.f. on X w.r.t. the equivalence relation (R). Hence, an element $\hat{x}$ of $\hat{X}$ is an equivalence class of c.f. on X w.r.t. (R).

2) Operations on $\hat{X}$
Multiplication by a scalar
Let $0 \neq \lambda \in \mathbb{K}$ and let $\hat{x}$ be a generic element of $\hat{X}$. For any $\mathcal{F}$ any representative of $\hat{x}$, we define $\lambda\hat{x}$ to be the equivalence class w.r.t. (R) of the the filter $\lambda\mathcal{F} := \{\lambda A : A \in \mathcal{F}\}$, i.e.

$$\lambda\hat{x} := \{\mathcal{G} \text{ c.f. on } X \,:\, \mathcal{G} \sim_R \lambda\mathcal{F}\}.$$

It is easy to check that this definition does not depend on the choice of the representative $\mathcal{F}$ of $\hat{x}$ (see Sheet 7, Exercise 1).
When $\lambda = 0$, we have $\lambda\hat{x} = \hat{o}$, where $\hat{o}$ is the equivalence class w.r.t. (R) of the filter of neighborhoods of the origin o in X (or, which is the same, of the Cauchy filter consisting of all the subsets of X which contain o).

Vector addition
Let $\hat{x}$ and $\hat{y}$ be two arbitrary elements of $\hat{X}$, and $\mathcal{F}$ (resp. $\mathcal{G}$) a representative of $\hat{x}$ (resp. $\hat{y}$). We define $\hat{x}+\hat{y}$ to be the equivalence class w.r.t. (R) of the the filter $\mathcal{F}+\mathcal{G} := \{C \subseteq X \,:\, A+B \subseteq C \text{ for some } A \in \mathcal{F},\, B \in \mathcal{G}\}$, i.e.

$$\hat{x}+\hat{y} := \{\mathcal{H} \text{ c.f. on } X \,:\, \mathcal{H} \sim_R \mathcal{F}+\mathcal{G}\}.$$

Note that this vector addition is well-defined because its definition does not depend on the choice of the representative $\mathcal{F}$ of $\hat{x}$ and $\mathcal{G}$ of $\hat{y}$ (see Sheet 7, Exercise 1).

3) Topology on $\hat{X}$
Let U be an arbitrary nbhood of the origin in X. Define

$$\hat{U} := \{\hat{x} \in \hat{X} : U \in \mathcal{F} \text{ for some } \mathcal{F} \in \hat{x}\}. \tag{2.3}$$

and consider the collection $\hat{\mathcal{B}} := \{\hat{U} : U \text{ nbhood of the origin in } X\}$. The filter generated by $\hat{\mathcal{B}}$ fulfills all the properties in Theorem 2.1.10 (see Sheet 7, Exercise 2) and therefore, it is the filter of nbhoods of the origin $\hat{o} \in \hat{X}$ w.r.t. to the unique topology on $\hat{X}$ compatible with the vector space structure defined in Step 2. Clearly, $\hat{\mathcal{B}}$ is a basis of nbhoods of the origin $\hat{o} \in \hat{X}$ w.r.t. to such a topology.

4) $\hat{X}$ is a Hausdorff t.v.s.
So far we have constructed a t.v.s. $\hat{X}$. In this step, we aim to prove that $\hat{X}$ is also Hausdorff. By Proposition 2.2.3, it is enough to show that for any $\hat{x} \in \hat{X}$ with $\hat{o} \neq \hat{x}$ there exists a nbhood $\hat{V}$ of the origin $\hat{o}$ in $\hat{X}$ s.t. $\hat{x} \notin \hat{V}$.
Since $\hat{o} \neq \hat{x}$, for any $\mathcal{F} \in \hat{x}$ and for any $\mathcal{F}_o \in \hat{o}$ we have $\mathcal{F} \not\sim_R \mathcal{F}_o$. Take $\mathcal{F}_0 := \{E \subseteq X : o \in E\}$, then the fact that $\mathcal{F} \not\sim_R \mathcal{F}_o$ means that there exists U nbhood of the origin in X s.t. $\forall\, A \in \mathcal{F}$ and $\forall\, A_o \in \mathcal{F}_o$ we have $A - A_o \not\subset U$. In particular, $\{o\} \in \mathcal{F}_o$ and so $\forall\, A \in \mathcal{F}$ we get $A \not\subset U$, which simply means that $U \notin \mathcal{F}$. By Theorem 2.1.10 applied to the t.v.s. X, we can always find another nbohood V of the origin in X s.t. $V + V \subset U$.
Claim: V does not belong to any representative of $\hat{x}$. This means, in view of the definition (2.3), that $\hat{x} \notin \hat{V}$. Hence, as observed at the beginning, the conclusion follows by Proposition 2.2.3.
Let us finally prove the claim. If $\mathcal{F}'$ is any representative of $\hat{x}$, then $\mathcal{F} \sim_R \mathcal{F}'$, i.e. $\exists\, A \in \mathcal{F}$ and $\exists\, A' \in \mathcal{F}'$ s.t. $A - A' \subset V$. Suppose that $V \in \mathcal{F}'$ then $A' \cap V \in \mathcal{F}'$ and so $A' \cap V \neq \emptyset$. Therefore, we clearly have $A - (A' \cap V) \subset V$ which implies

$$A \subset V + (A' \cap V) \subset V + V \subset U.$$

Since $A \in \mathcal{F}$, this proves that $U \in \mathcal{F}$ which is a contradiction. Then $V \notin \mathcal{F}'$ for all $\mathcal{F}' \in \hat{x}$ that is exactly our claim.

5) Existence of $i : X \to \hat{X}$
We define the image of a point $x \in X$ under the mapping $i : X \to \hat{X}$ to be the equivalence class w.r.t. (R) of the filter $\mathcal{F}(x)$ of neighborhoods of x in X, i.e.

$$\forall\, x \in X,\ \ i(x) := \{\mathcal{F} \text{ c.f. on } X : \mathcal{F} \sim_R \mathcal{F}(x)\}.$$

Note that the following properties hold.

Lemma 2.5.16.

a) Two c.f. filters on X converging to the same point are equivalent w.r.t. (R)
b) If two c.f. filters $\mathcal{F}$ and $\mathcal{F}'$ on X are s.t. $\mathcal{F} \sim_R \mathcal{F}'$ and $\mathcal{F}'$ converges to $x \in X$ then also $\mathcal{F}$ converges to x.

Proof. (Sheet 7, Exercise 3) □

The previous lemma clearly proves that

$$i(x) \equiv \{\mathcal{F} \text{ c.f. on } X : \mathcal{F} \to x)\}.$$

6) i is an injective linear homeomorphism (i.e. (a) holds)
<u>i is injective</u>
(see Sheet 7, Exercise 4).

<u>i is linear</u>
(see Sheet 7, Exercise 4).

<u>i is a homemorphism</u>
We aim to show that i is both open and continuous on X.

To prove that i is open, we need to show that for any nbhood U of the origin in X the image $i(U)$ is a nbhood of the origin in $i(X)$ endowed with the subset topology induced by the topology on $\hat{X}$. Therefore, it suffices to show that for any nbhood U of the origin in X there exists U_1 nbhood of the origin in X s.t.

$$\hat{U_1} \cap i(X) \subseteq i(U) \tag{2.4}$$

where $\hat{U_1}$ is defined as in (2.3).

To show the continuity of i, we need to prove that for any nbhood $\hat{V}$ of the origin in $i(X)$ the preimage $i^{-1}(\hat{V})$ is a nbhood of the origin in X. Now any nbhood of the origin in $i(X)$ is of the form $\hat{U_1} \cap i(X)$ for some U_1 nbhood of the origin in X. Therefore, it is enough to show that for any U_1

nbhood of the origin in X there exists another U nbhood of the origin in X s.t. $U \subseteq i^{-1}(\hat{U}_1 \cap i(X))$ i.e.

$$i(U) \subseteq \hat{U}_1 \cap i(X) \tag{2.5}$$

In order to prove (2.4) and (2.5), we shall prove the following:

$$i(\mathring{V}) \subseteq \hat{V} \cap i(X) \subseteq i(\overline{V}), \quad \forall V \text{ nbhood of the origin in } X. \tag{2.6}$$

Indeed, if (2.6) holds then the first inclusion immediately shows (2.5) (for any U_1 nbhood of the origin in X take $U := \mathring{U}_1$ and apply the first inclusion of (2.6) to $V = U_1$). Moreover, (2.4) follows by combining the fact that for any nbhood U of the origin in X exists another nbhood U_1 of the origin in X s.t. $\overline{U_1} = U_1 \subseteq U$ (c.f. Sheet 3, Ex3-a)) together with the second inclusion of (2.6) (applied to U_1).

It remains to prove that (2.6) holds. Let V be any nbhood of the origin in X, then for any $x \in \mathring{V}$ we clearly have that V is a nbhood of x, which means that V belongs to a representative of $i(x)$, i.e. $i(x) \in \hat{V}$. Hence, $i(\mathring{V}) \subseteq \hat{V} \cap i(X)$. Now take $\hat{y} \in \hat{V} \cap i(X)$, i.e. $\hat{y} = i(x)$ for some $x \in X$ s.t. $i(x) \in \hat{V}$. Then, by definition (2.3), we have that $V \in \mathcal{F}$ for some $\mathcal{F} \in i(x)$ or in other words that V belongs to some filter $\mathcal{F}$ converging to x. Let W be another nbhood of the origin in X then $W + x$ is a nbhood of x in X and so $W + x \in \mathcal{F}$ (since $\mathcal{F} \to x$). Hence, $V \cap (W + x) \in \mathcal{F}$ which implies that $V \cap (W + x) \neq \emptyset$ i.e. $x \in \overline{V}$. This means that $\hat{y} = i(x) \in i(\overline{V})$ which proves $\hat{V} \cap i(X) \subseteq i(\overline{V})$.

7) $\overline{i(X)} = \hat{X}$ (i.e. (b) holds)

Let $\hat{x_o} \in \hat{X}$ and let N be any nbhood of $\hat{x_o}$ in $\hat{X}$. It suffices to consider the neighborhoods N of the form $\hat{U} + \hat{x_0}$ where $\hat{U}$ is defined by (2.3) for some U nbhood of the origin in X. We aim to prove that $(\hat{U} + \hat{x_o}) \cap i(X) \neq \emptyset$.

By Theorem 2.1.10, we know that for any U nbhood of the origin in X there exists V nbhood of the origin in X s.t. $V + V \subset U$. Let $\mathcal{F}_o$ be a representative of $\hat{x_0}$, then $\mathcal{F}_o$ is a c.f. on X and so there exists $A_o \in \mathcal{F}_o$ s.t. $A_o - A_o \subset V$. Fix an element $x \in A_o$. Then we get:

$$(V + x) - A_o \subset V + A_o - A_o \subset V + V \subset U. \tag{2.7}$$

Since $V + x$ is a nbhood of x in X, $V + x$ belongs to any Cauchy filter $\mathcal{F}$ converging to x and so $V + x \in \mathcal{F}$ for any $\mathcal{F} \in i(x)$. Then $(V + x) - A_o \in \mathcal{F} - \mathcal{F}_o$ and so (2.7) gives $U \in \mathcal{F} - \mathcal{F}_o$ i.e. $i(x) - \hat{x_o} \in \hat{U}$. Hence, we found that there exists $x \in X$ s.t. $i(x) \in \hat{U} + \hat{x_o}$ which gives the conclusion.

8) $\hat{X}$ is complete
Let $\hat{\mathcal{F}}$ be a Cauchy filter on $\hat{X}$. We aim to prove that there exists an element $\hat{x} \in \hat{X}$ s.t. $\hat{\mathcal{F}} \to \hat{x}$.

Consider the filter

$$\hat{\mathcal{F}}' := \{\hat{G} \subset \hat{X} : \hat{M} + \hat{U} \subset \hat{G} \text{ for some } \hat{M} \in \hat{\mathcal{F}} \text{ and } \hat{U} \text{ nbhood of the origin in } \hat{X}\}.$$

Note that $\hat{\mathcal{F}}' \subset \hat{\mathcal{F}}$ and $\hat{\mathcal{F}}'$ is also a Cauchy filter on $\hat{X}$. In fact, since $\hat{X}$ is a t.v.s., for any $\hat{U}$ nbhood of the origin in $\hat{X}$ there exists $\hat{V}_0$ balanced nbhood of the origin in $\hat{X}$ s.t. $\hat{V}_0 + \hat{V}_0 + \hat{V}_0 \subset U$. Take $\hat{V} := \frac{1}{3}\hat{V}_0$ which is also a nbhood of the origin in $\hat{X}$, then

$$\hat{V} + \hat{V} - \hat{V} \subset \hat{V}_0 + \hat{V}_0 + \hat{V}_0 \subset U.$$

Since $\hat{F}$ is a Cauchy filter, there exists $\hat{M} \in \hat{F}$ s.t. $\hat{M} - \hat{M} \subset \hat{V}$. Then

$$(\hat{M} + \hat{V}) - (\hat{M} + \hat{V}) \subset \hat{V} + \hat{V} - \hat{V} \subset U$$

Now let us consider the family of subsets of $i(X)$ given by

$$\mathcal{F}' := \{\hat{A} \cap i(X) : \hat{A} \in \hat{F}'\}.$$

It is possible to prove that $\mathcal{F}'$ is a filter on $i(X)$ and actually a Cauchy filter (see Sheet 7, Exercise 5). Moreover, since we proved that i is a topological isomorphism between X and $i(X)$, we have that $i^{-1}(\mathcal{F}')$ is a Cauchy filter on X. Take

$$\hat{x} := \{\mathcal{F} \text{ c.f. on } X : \mathcal{F} \sim_R i^{-1}(\mathcal{F}')\}.$$

Then $\hat{F}$ converges to $\hat{x}$ (see Sheet 7, Exercise 5).

9) Proof of the universal property (i.e. (c) and (II))
We can now identify X with $i(X)$ and so regard X as a dense linear subspace of $\hat{X}$. Since $f : X \to Y$ is continuous and linear by assumption, it is also uniformly continuous by Proposition 2.5.13. Then applying Theorem 2.5.14 with X replaced by $\hat{X}$ and A by X we get both the properties (c) and (II).

10) Uniqueness of $\hat{X}$ up to isomorphism (proof of (I))
Since by assumption $\hat{X}_1$ is a complete Hausdorff t.v.s. and $i_1 : X \to \hat{X}_1$ is a topological monomorphism (in particular i_1 is a continuous linear mapping), we have by (c) that there exists a unique continuous linear map $\hat{i_1}$ s.t. $\hat{i_1}(i(x)) = i_1(x)$ for any $x \in X$. Let us define $j := \hat{i_1}$. On the other hand, let us define $f : i_1(X) \to \hat{X}$ by $f(i_1(x)) = i(x)$ for any $x \in X$. Since f is

clearly linear and continuous and $i_1(X)$ is a linear subspace of $\hat{X}$, f is uniformly continuous and so by Theorem 2.5.14 we get that there exists a unique $\hat{f} : \hat{X_1} \to \hat{X}$ continuous and linear s.t. $\hat{f}(i_1(x)) = f(i_1(x))$ for any $x \in X$. Using the density of $i(X)$ in $\hat{X}$, the density of $i_1(X)$ in $\hat{X_1}$ and the continuity of the mappings involved, it is easy to check that

$$\bar{f}(j(\hat{x})) = \hat{x} \ \forall \, \hat{x} \in \hat{X}$$

and that

$$j(\bar{f}(\hat{x_1})) = \hat{x_1} \ \forall \, , \hat{x_1} \in \hat{X_1}.$$

This means that j and f are the inverse of each other and that both are isomorphisms.

□