

Chapter 3

Finite dimensional topological vector spaces

3.1 Finite dimensional Hausdorff t.v.s.

Let X be a vector space over the field \mathbb{K} of real or complex numbers. We know from linear algebra that the (algebraic) dimension of X , denoted by $\dim(X)$, is the cardinality of a basis of X . If $\dim(X)$ is finite, we say that X is *finite dimensional* otherwise X is *infinite dimensional*. In this section we are going to focus on finite dimensional vector spaces.

Let $\{e_1, \dots, e_d\}$ be a basis of X , i.e. $\dim(X) = d$. Given any vector $x \in X$ there exist unique $x_1, \dots, x_d \in \mathbb{K}$ s.t. $x = x_1e_1 + \dots + x_de_d$. This can be precisely expressed by saying that the mapping

$$\begin{array}{ccc} \mathbb{K}^d & \rightarrow & X \\ (x_1, \dots, x_d) & \mapsto & x_1e_1 + \dots + x_de_d \end{array}$$

is an algebraic isomorphism (i.e. linear and bijective) between X and \mathbb{K}^d . In other words: *If X is a finite dimensional vector space then X is algebraically isomorphic to $\mathbb{K}^{\dim(X)}$.*

If now we give to X the t.v.s. structure and we consider \mathbb{K} endowed with the euclidean topology, then it is natural to ask if such an algebraic isomorphism is by any chance a topological one, i.e. if it preserves the t.v.s. structure. The following theorem shows that if X is a finite dimensional Hausdorff t.v.s. then the answer is yes: X is topologically isomorphic to $\mathbb{K}^{\dim(X)}$. It is worth to observe that usually in applications we deal always with Hausdorff t.v.s., therefore it makes sense to mainly focus on them.

Theorem 3.1.1. *Let X be a finite dimensional Hausdorff t.v.s. over \mathbb{K} (where \mathbb{K} is endowed with the euclidean topology). Then:*

a) *X is topologically isomorphic to \mathbb{K}^d , where $d = \dim(X)$.*

- b) Every linear functional on X is continuous.
- c) Every linear map of X into any t.v.s. Y is continuous.

Before proving the theorem let us recall some lemmas about the continuity of linear functionals on t.v.s..

Lemma 3.1.2.

Let X be a t.v.s. over \mathbb{K} and $v \in X$. Then the following mapping is continuous.

$$\begin{aligned} \varphi_v : \mathbb{K} &\rightarrow X \\ \xi &\mapsto \xi v. \end{aligned}$$

Proof. For any $\xi \in \mathbb{K}$, we have $\varphi_v(\xi) = M(\psi_v(\xi))$, where $\psi_v : \mathbb{K} \rightarrow \mathbb{K} \times X$ given by $\psi(\xi) := (\xi, v)$ is clearly continuous by definition of product topology and $M : \mathbb{K} \times X \rightarrow X$ is the scalar multiplication in the t.v.s. X which is continuous by definition of t.v.s.. Hence, φ_v is continuous as composition of continuous mappings. \square

Lemma 3.1.3. Let X be a t.v.s. over \mathbb{K} and L a linear functional on X . Assume $L(x) \neq 0$ for some $x \in X$. Then the following are equivalent:

- a) L is continuous.
- b) The null space $\text{Ker}(L)$ is closed in X
- c) $\text{Ker}(L)$ is not dense in X .
- d) L is bounded in some neighbourhood of the origin in X .

Proof. (see Sheet 4, Exercise 4)

Proof. of Theorem 3.1.1

Let $\{e_1, \dots, e_d\}$ be a basis of X and let us consider the mapping

$$\begin{aligned} \varphi : \mathbb{K}^d &\rightarrow X \\ (x_1, \dots, x_d) &\mapsto x_1 e_1 + \dots + x_d e_d. \end{aligned}$$

As noted above, this is an algebraic isomorphism. Therefore, to conclude a) it remains to prove that φ is also a homeomorphism.

Step 1: φ is continuous.

When $d = 1$, we simply have $\varphi \equiv \varphi_{e_1}$ and so we are done by Lemma 3.1.2. When $d > 1$, for any $(x_1, \dots, x_d) \in \mathbb{K}^d$ we can write: $\varphi(x_1, \dots, x_d) = A(\varphi_{e_1}(x_1), \dots, \varphi_{e_d}(x_d)) = A((\varphi_{e_1} \times \dots \times \varphi_{e_d})(x_1, \dots, x_d))$ where each φ_{e_j} is defined as above and $A : X \times X \rightarrow X$ is the vector addition in the t.v.s. X . Hence, φ is continuous as composition of continuous mappings.

Step 2: φ is open and b) holds.

We prove this step by induction on the dimension $\dim(X)$ of X .

For $\dim(X) = 1$, it is easy to see that φ is open, i.e. that the inverse of φ :

$$\begin{aligned} \varphi^{-1} : \quad X &\rightarrow \mathbb{K} \\ x = \xi e_1 &\mapsto \xi \end{aligned}$$

is continuous. Indeed, we have that

$$\text{Ker}(\varphi^{-1}) = \{x \in X : \varphi^{-1}(x) = 0\} = \{\xi e_1 \in X : \xi = 0\} = \{o\},$$

which is closed in X , since X is Hausdorff. Hence, by Lemma 3.1.3, φ^{-1} is continuous. This implies that b) holds. In fact, if L is a non-identically zero functional on X (when $L \equiv 0$, there is nothing to prove), then there exists a $o \neq \tilde{x} \in X$ s.t. $L(\tilde{x}) \neq 0$. W.l.o.g. we can assume $L(\tilde{x}) = 1$. Now for any $x \in X$, since $\dim(X) = 1$, we have that $x = \xi \tilde{x}$ for some $\xi \in \mathbb{K}$ and so $L(x) = \xi L(\tilde{x}) = \xi$. Hence, $L \equiv \varphi^{-1}$ which we proved to be continuous.

Suppose now that both a) and b) hold for $\dim(X) \leq d-1$. Let us first show that b) holds when $\dim(X) = d$. Let L be a non-identically zero functional on X (when $L \equiv 0$, there is nothing to prove), then there exists a $o \neq \tilde{x} \in X$ s.t. $L(\tilde{x}) \neq 0$. W.l.o.g. we can assume $L(\tilde{x}) = 1$. Note that for any $x \in X$ the element $x - \tilde{x}L(x) \in \text{Ker}(L)$. Therefore, if we take the canonical mapping $\phi : X \rightarrow X/\text{Ker}(L)$ then $\phi(x) = \phi(\tilde{x}L(x)) = L(x)\phi(\tilde{x})$ for any $x \in X$. This means that $X/\text{Ker}(L) = \text{span}\{\phi(\tilde{x})\}$ i.e. $\dim(X/\text{Ker}(L)) = 1$. Hence, $\dim(\text{Ker}(L)) = d-1$ and so by inductive assumption $\text{Ker}(L)$ is topologically isomorphic to \mathbb{K}^{d-1} . This implies that $\text{Ker}(L)$ is a complete subspace of X . Then, by Proposition 2.5.8-a), $\text{Ker}(L)$ is closed in X and so by Lemma 3.1.3 we get L is continuous. By induction, we can conclude that b) holds for any dimension $d \in \mathbb{N}$.

This immediately implies that a) holds for any dimension $d \in \mathbb{N}$. In fact, we just need to show that for any dimension $d \in \mathbb{N}$ the mapping

$$\begin{aligned} \varphi^{-1} : \quad X &\rightarrow \mathbb{K}^d \\ x = \sum_{j=1}^d x_j e_j &\mapsto (x_1, \dots, x_d) \end{aligned}$$

is continuous. Now for any $x = \sum_{j=1}^d x_j e_j \in X$ we can write $\varphi^{-1}(x) = (L_1(x), \dots, L_d(x))$, where for any $j \in \{1, \dots, d\}$ we define $L_j : X \rightarrow \mathbb{K}$ by $L_j(x) := x_j e_j$. Since b) holds for any dimension, we know that each L_j is continuous and so φ^{-1} is continuous.

Step 3: The statement c) holds.

Let $f : X \rightarrow Y$ be linear and $\{e_1, \dots, e_d\}$ be a basis of X . For any $j \in \{1, \dots, d\}$ we define $b_j := f(e_j) \in Y$. Hence, for any $x = \sum_{j=1}^d x_j e_j \in X$ we

have $f(x) = f(\sum_{j=1}^d x_j e_j) = \sum_{j=1}^d x_j b_j$. We can rewrite f as composition of continuous maps i.e. $f(x) = A((\varphi_{b_1} \times \dots \times \varphi_{b_d})(\varphi^{-1}(x)))$ where:

- φ^{-1} is continuous by a)
- each φ_{b_j} is continuous by Lemma 3.1.2
- A is the vector addition on X and so it is continuous since X is a t.v.s..

Hence, f is continuous. □

Corollary 3.1.4 (Tychonoff theorem). *Let $d \in \mathbb{N}$. The only topology that makes \mathbb{K}^d a Hausdorff t.v.s. is the euclidean topology. Equivalently, on a finite dimensional vector space there is a unique topology that makes it into a Hausdorff t.v.s..*

Proof. We already know that \mathbb{K}^d endowed with the euclidean topology τ_e is a Hausdorff t.v.s. of dimension d . Let us consider another topology τ on \mathbb{K}^d s.t. (\mathbb{K}^d, τ) is also Hausdorff t.v.s.. Then Theorem 3.1.1-a) ensures that the identity map between (\mathbb{K}^d, τ_e) and (\mathbb{K}^d, τ) is a topological isomorphism. Hence, as observed at the end of Section 1.1.4 p.12, we get that $\tau \equiv \tau_e$. □

Corollary 3.1.5. *Every finite dimensional Hausdorff t.v.s. is complete.*

Proof. Let X be a Hausdorff t.v.s with $\dim(X) = d < \infty$. Then, by Theorem 3.1.1-a), X is topologically isomorphic to \mathbb{K}^d endowed with the euclidean topology. Since the latter is a complete Hausdorff t.v.s., so is X . □

Corollary 3.1.6. *Every finite dimensional linear subspace of a Hausdorff t.v.s. is closed.*

Proof. Let S be a linear subspace of a Hausdorff t.v.s. (X, τ) and assume that $\dim(S) = d < \infty$. Then S endowed with the subspace topology induced by τ is itself a Hausdorff t.v.s. (see Sheet 5, Exercise 2). Hence, by Corollary 3.1.5 S is complete and therefore closed by Proposition 2.5.8-a). □

3.2 Connection between local compactness and finite dimensionality

By the Heine-Borel property (a subset of \mathbb{K}^d is closed and bounded iff it is compact), \mathbb{K}^d has a basis of compact neighborhoods of the origin (i.e. the closed balls centered at the origin in \mathbb{K}^d). Thus the origin, and consequently every point, of a finite dimensional Hausdorff t.v.s. has a basis of neighborhoods consisting of compact subsets. This means that a *finite dimensional Hausdorff*

t.v.s. is always locally compact. Actually also the converse is true and gives the following beautiful characterization of finite dimensional Hausdorff t.v.s due to F. Riesz.

Theorem 3.2.1. *A Hausdorff t.v.s. is locally compact if and only if it is finite dimensional.*

For convenience let us recall the notions of compactness and local compactness for topological spaces before proving the theorem.

Definition 3.2.2. *A topological space X is compact if every open covering of X contains a finite subcovering. i.e. for any arbitrary collection $\{U_i\}_{i \in I}$ of open subsets of X s.t. $X \subseteq \cup_{i \in I} U_i$ there exists a finite subset J of I s.t. $X \subseteq \cup_{i \in J} U_i$.*

Definition 3.2.3. *A topological space X is locally compact if every point of X has a base of compact neighbourhoods.*

Just a small side remark: It is possible to show that every compact Hausdorff space is also locally compact but there exist locally compact spaces that are not compact such as:

- \mathbb{K}^d with the euclidean topology
- any infinite set endowed with the discrete topology.

Indeed, any set X with the discrete topology is locally compact, because for any $x \in X$ a basis of neighbourhoods of x in the discrete topology is given just $\{x\}$ which is open and also compact. However, if X is infinite then it is not compact. In fact, if we take the infinite open covering \mathcal{S} of X given by all the singletons of its points, then any finite subcovering of \mathcal{S} will not contain at least one point of X .

Proof. of Theorem 3.2.1

Let X be a locally compact Hausdorff t.v.s., and K a compact neighborhood of o in X . As K is compact and as $\frac{1}{2}K$ is a neighborhood of the origin (see Theorem 2.1.10-3), there is a finite family of points $x_1, \dots, x_r \in X$ s.t.

$$K \subseteq \bigcup_{i=1}^r (x_i + \frac{1}{2}K).$$

Let $M := \text{span}\{x_1, \dots, x_r\}$. Then M is a linear subspace of X and $\dim(M) < \infty$ is finite, hence M is closed in X by Corollary 3.1.6. Therefore, the quotient space X/M is Hausdorff t.v.s. by Proposition 2.3.5.

Let $\phi : X \rightarrow X/M$ be the canonical mapping. As $K \subseteq M + \frac{1}{2}K$, we have $\phi(K) \subseteq \phi(M) + \phi(\frac{1}{2}K) = \frac{1}{2}\phi(K)$, i.e. $2\phi(K) \subseteq \phi(K)$. By iterating we get

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$\phi(2^n K) \subseteq \phi(K)$ for any $n \in \mathbb{N}$. As K is absorbing (see Theorem 2.1.10-5), we have $X = \bigcup_{n=1}^{\infty} 2^n K$. Thus

$$X/M = \phi(X) = \bigcup_{n=1}^{\infty} \phi(2^n K) \subseteq \phi(K).$$

Since ϕ is continuous and the continuous image of a compact set is compact, we get that $\phi(K)$ is compact. Thus X/M is a Hausdorff t.v.s. which is compact. We claim that X/M must be of zero dimension, i.e. reduced to one point. This concludes the proof because it implies $\dim(X) = \dim(M) < \infty$.

Let us prove the claim by contradiction. Suppose $\dim(X/M) > 0$ then X/M contains a subset of the form $\mathbb{R}\bar{x}$ for some $\bar{o} \neq \bar{x} \in X/M$. Since such a subset is closed and X/M is compact, $\mathbb{R}\bar{x}$ is also compact which is a contradiction. \square