Chapter 3

Finite dimensional topological vector spaces

3.1 Finite dimensional Hausdorff t.v.s.

Let \( X \) be a vector space over the field \( \mathbb{K} \) of real or complex numbers. We know from linear algebra that the (algebraic) dimension of \( X \), denoted by \( \dim(X) \), is the cardinality of a basis of \( X \). If \( \dim(X) \) is finite, we say that \( X \) is finite dimensional otherwise \( X \) is infinite dimensional. In this section we are going to focus on finite dimensional vector spaces.

Let \( \{e_1, \ldots, e_d\} \) be a basis of \( X \), i.e. \( \dim(X) = d \). Given any vector \( x \in X \) there exist unique \( x_1, \ldots, x_d \in \mathbb{K} \) s.t. \( x = x_1 e_1 + \cdots x_d e_d \). This can be precisely expressed by saying that the mapping

\[
\mathbb{K}^d \rightarrow X
\]

\[
(x_1, \ldots, x_d) \mapsto x_1 e_1 + \cdots x_d e_d
\]

is an algebraic isomorphism (i.e. linear and bijective) between \( X \) and \( \mathbb{K}^d \). In other words: If \( X \) is a finite dimensional vector space then \( X \) is algebraically isomorphic to \( \mathbb{K}^{\dim(X)} \).

If now we give to \( X \) the t.v.s. structure and we consider \( \mathbb{K} \) endowed with the euclidean topology, then it is natural to ask if such an algebraic isomorphism is by any chance a topological one, i.e. if it preserves the t.v.s. structure. The following theorem shows that if \( X \) is a finite dimensional Hausdorff t.v.s. then the answer is yes: \( X \) is topologically isomorphic to \( \mathbb{K}^{\dim(X)} \). It is worth to observe that usually in applications we deal always with Hausdorff t.v.s., therefore it makes sense to mainly focus on them.

**Theorem 3.1.1.** Let \( X \) be a finite dimensional Hausdorff t.v.s. over \( \mathbb{K} \) (where \( \mathbb{K} \) is endowed with the euclidean topology). Then:

a) \( X \) is topologically isomorphic to \( \mathbb{K}^d \), where \( d = \dim(X) \).
b) Every linear functional on $X$ is continuous.

c) Every linear map of $X$ into any t.v.s. $Y$ is continuous.

Before proving the theorem let us recall some lemmas about the continuity of linear functionals on t.v.s.

**Lemma 3.1.2.**

Let $X$ be a t.v.s over $\mathbb{K}$ and $v \in X$. Then the following mapping is continuous.

$$\varphi_v : \mathbb{K} \to X \quad \xi \mapsto \xi v.$$ 

**Proof.** For any $\xi \in \mathbb{K}$, we have $\varphi_v(\xi) = M(\psi_v(\xi))$, where $\psi_v : \mathbb{K} \to \mathbb{K} \times X$ given by $\psi(\xi) := (\xi, v)$ is clearly continuous by definition of product topology and $M : \mathbb{K} \times X \to X$ is the scalar multiplication in the t.v.s. $X$ which is continuous by definition of t.v.s. Hence, $\varphi_v$ is continuous as composition of continuous mappings.

**Lemma 3.1.3.** Let $X$ be a t.v.s over $\mathbb{K}$ and $L$ a linear functional on $X$. Assume $L(x) \neq 0$ for some $x \in X$. Then the following are equivalent:

a) $L$ is continuous.

b) The null space $\ker(L)$ is closed in $X$

c) $\ker(L)$ is not dense in $X$.

d) $L$ is bounded in some neighbourhood of the origin in $X$.

**Proof.** (see Sheet 4, Exercise 4)

**Proof. of Theorem 3.1.1**

Let $\{e_1, \ldots, e_d\} \subseteq X$ be a basis of $X$ and let us consider the mapping

$$\varphi : \mathbb{K}^d \to X \quad (x_1, \ldots, x_d) \mapsto x_1 e_1 + \cdots + x_d e_d.$$ 

As noted above, this is an algebraic isomorphism. Therefore, to conclude a) it remains to prove that $\varphi$ is also a homeomorphism.

**Step 1:** $\varphi$ is continuous.

When $d = 1$, we simply have $\varphi \equiv \varphi_{e_1}$ and so we are done by Lemma 3.1.2. When $d > 1$, for any $(x_1, \ldots, x_d) \in \mathbb{K}^d$ we can write: $\varphi(x_1, \ldots, x_d) = A(\varphi_{e_1}(x_1), \ldots, \varphi_{e_d}(x_d))$ $= A((\varphi_{e_1} \times \cdots \times \varphi_{e_d})(x_1, \ldots, x_d))$ where each $\varphi_{e_j}$ is defined as above and $A : X \times X \to X$ is the vector addition in the t.v.s. $X$. Hence, $\varphi$ is continuous as composition of continuous mappings.
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Step 2: φ is open and b) holds.
We prove this step by induction on the dimension \( \dim(X) \) of \( X \).
For \( \dim(X) = 1 \), it is easy to see that \( \varphi \) is open, i.e. that the inverse of \( \varphi \):
\[
\varphi^{-1} : X \rightarrow \mathbb{K} \\
x = \xi e_1 \mapsto \xi
\]
is continuous. Indeed, we have that
\[
\ker(\varphi^{-1}) = \{ x \in X : \varphi^{-1}(x) = 0 \} = \{ \xi e_1 \in X : \xi = 0 \} = \{0\},
\]
which is closed in \( X \), since \( X \) is Hausdorff. Hence, by Lemma 3.1.3, \( \varphi^{-1} \) is continuous.
This implies that b) holds. In fact, if \( L \) is a non-identically zero functional on \( X \) (when \( L \equiv 0 \), there is nothing to prove), then there exists a \( \in X \) such that \( L(\in) \neq 0 \). W.l.o.g. we can assume \( L(\in) = 1 \). Now for any \( x \in X \), since \( \dim(X) = 1 \), we have that \( x = \xi \in \) for some \( \xi \in \mathbb{K} \) and so \( L(x) = \xi L(\in) = \xi \). Hence, \( L \equiv \varphi^{-1} \) which we proved to be continuous.

Suppose now that both a) and b) hold for \( \dim(X) \leq d-1 \). Let us first show that b) holds when \( \dim(X) = d \). Let \( L \) be a non-identically zero functional on \( X \) (when \( L \equiv 0 \), there is nothing to prove), then there exists a \( \in X \) such that \( L(\in) \neq 0 \). W.l.o.g. we can assume \( L(\in) = 1 \). Note that for any \( x \in X \) the element \( x - \in L(x) \in \ker(L) \). Therefore, if we take the canonical mapping \( \phi : X \rightarrow X/\ker(L) \) then \( \phi(x) = \phi(\in L(x)) = L(x) \phi(\in) \) for any \( x \in X \).
This means that \( X/\ker(L) = \text{span}\{\phi(\in)\} \) i.e. \( \dim(X/\ker(L)) = 1 \). Hence, \( \dim(\ker(L)) = d-1 \) and so by inductive assumption \( \ker(L) \) is topologically isomorphic to \( \mathbb{K}^{d-1} \). This implies that \( \ker(L) \) is a complete subspace of \( X \). Then, by Proposition 2.5.8-a), \( \ker(L) \) is closed in \( X \) and so by Lemma 3.1.3 we get \( L \) is continuous. By induction, we can conclude that b) holds for any dimension \( d \in \mathbb{N} \).

This immediately implies that a) holds for any dimension \( d \in \mathbb{N} \). In fact, we just need to show that for any dimension \( d \in \mathbb{N} \) the mapping
\[
\varphi^{-1} : X \rightarrow \mathbb{K}^d \\
x = \sum_{j=1}^d x_j e_j \mapsto (x_1, \ldots, x_d)
\]
is continuous. Now for any \( x = \sum_{j=1}^d x_j e_j \in X \) we can write \( \varphi^{-1}(x) = (L_1(x), \ldots, L_d(x)) \), where for any \( j \in \{1, \ldots, d\} \) we define \( L_j : X \rightarrow \mathbb{K} \) by \( L_j(x) := x_j e_j \). Since b) holds for any dimension, we know that each \( L_j \) is continuous and so \( \varphi^{-1} \) is continuous.

Step 3: The statement c) holds.
Let \( f : X \rightarrow Y \) be linear and \( \{e_1, \ldots, e_d\} \) be a basis of \( X \). For any \( j \in \{1, \ldots, d\} \) we define \( b_j := f(e_j) \in Y \). Hence, for any \( x = \sum_{j=1}^d x_j e_j \in X \) we
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have \( f(x) = f(\sum_{j=1}^{d} x_j e_j) = \sum_{j=1}^{d} x_j b_j \). We can rewrite \( f \) as composition of continuous maps i.e. \( f(x) = A((\varphi_{b_1} \times \cdots \times \varphi_{b_d})(\varphi^{-1}(x))) \) where:
- \( \varphi^{-1} \) is continuous by a)
- each \( \varphi_{b_j} \) is continuous by Lemma 3.1.2
- \( A \) is the vector addition on \( X \) and so it is continuous since \( X \) is a t.v.s.
Hence, \( f \) is continuous.

**Corollary 3.1.4** (Tychonoff theorem). Let \( d \in \mathbb{N} \). The only topology that makes \( \mathbb{K}^d \) a Hausdorff t.v.s. is the euclidean topology. Equivalently, on a finite dimensional vector space there is a unique topology that makes it into a Hausdorff t.v.s.

**Proof.** We already know that \( \mathbb{K}^d \) endowed with the euclidean topology \( \tau_e \) is a Hausdorff t.v.s. of dimension \( d \). Let us consider another topology \( \tau \) on \( \mathbb{K}^d \) s.t. \( (\mathbb{K}^d, \tau) \) is also Hausdorff t.v.s.. Then Theorem 3.1.1-a) ensures that the identity map between \( (\mathbb{K}^d, \tau_e) \) and \( (\mathbb{K}^d, \tau) \) is a topological isomorphism. Hence, as observed at the end of Section 1.1.4 p.12, we get that \( \tau \equiv \tau_e \). □

**Corollary 3.1.5.** Every finite dimensional Hausdorff t.v.s. is complete.

**Proof.** Let \( X \) be a Hausdorff t.v.s with \( \dim(X) = d < \infty \). Then, by Theorem 3.1.1-a), \( X \) is topologically isomorphic to \( \mathbb{K}^d \) endowed with the euclidean topology. Since the latter is a complete Hausdorff t.v.s., so is \( X \). □

**Corollary 3.1.6.** Every finite dimensional linear subspace of a Hausdorff t.v.s. is closed.

**Proof.** Let \( S \) be a linear subspace of a Hausdorff t.v.s. \( (X, \tau) \) and assume that \( \dim(S) = d < \infty \). Then \( S \) endowed with the subspace topology induced by \( \tau \) is itself a Hausdorff t.v.s. (see Sheet 5, Exercise 2). Hence, by Corollary 3.1.5 \( S \) is complete and therefore closed by Proposition 2.5.8-a). □

### 3.2 Connection between local compactness and finite dimensionality

By the Heine-Borel property (a subset of \( \mathbb{K}^d \) is closed and bounded iff it is compact), \( \mathbb{K}^d \) has a basis of compact neighborhoods of the origin (i.e. the closed balls centered at the origin in \( \mathbb{K}^d \)). Thus the origin, and consequently every point, of a finite dimensional Hausdorff t.v.s. has a basis of neighborhoods consisting of compact subsets. This means that a finite dimensional Hausdorff...
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t.v.s. is always locally compact. Actually also the converse is true and gives the following beautiful characterization of finite dimensional Hausdorff t.v.s due to F. Riesz.

**Theorem 3.2.1.** A Hausdorff t.v.s. is locally compact if and only if it is finite dimensional.

For convenience let us recall the notions of compactness and local compactness for topological spaces before proving the theorem.

**Definition 3.2.2.** A topological space $X$ is compact if every open covering of $X$ contains a finite subcovering. i.e. for any arbitrary collection $\{U_i\}_{i \in I}$ of open subsets of $X$ s.t. $X \subseteq \cup_{i \in I} U_i$ there exists a finite subset $J$ of $I$ s.t. $X \subseteq \cup_{i \in J} U_i$.

**Definition 3.2.3.** A topological space $X$ is locally compact if every point of $X$ has a base of compact neighbourhoods.

Just a small side remark: It is possible to show that every compact Hausdorff space is also locally compact but there exist locally compact spaces that are not compact such as:

- $\mathbb{K}^d$ with the euclidean topology
- any infinite set endowed with the discrete topology.

Indeed, any set $X$ with the discrete topology is locally compact, because for any $x \in X$ a basis of neighbourhoods of $x$ in the discrete topology is given just $\{x\}$ which is open and also compact. However, if $X$ is infinite then it is not compact. In fact, if we take the infinite open covering $\mathcal{S}$ of $X$ given by all the singletons of its points, then any finite subcovering of $\mathcal{S}$ will not contain at least one point of $X$.

**Proof.** of Theorem 3.2.1

Let $X$ be a locally compact Hausdorff t.v.s., and $K$ a compact neighborhood of $o$ in $X$. As $K$ is compact and as $\frac{1}{2}K$ is a neighborhood of the origin (see Theorem 2.1.10-3), there is a finite family of points $x_1, \ldots, x_r \in X$ s.t.

$$K \subseteq \bigcup_{i=1}^{r}(x_i + \frac{1}{2}K).$$

Let $M := \text{span}\{x_1, \ldots, x_r\}$. Then $M$ is a linear subspace of $X$ and $\dim(M) < \infty$ is finite, hence $M$ is closed in $X$ by Corollary 3.1.6. Therefore, the quotient space $X/M$ is Hausdorff t.v.s. by Proposition 2.3.5.

Let $\phi : X \to X/M$ be the canonical mapping. As $K \subseteq M + \frac{1}{2}K$, we have $\phi(K) \subseteq \phi(M) + \phi(\frac{1}{2}K) = \frac{1}{2}\phi(K)$, i.e. $2\phi(K) \subseteq \phi(K)$. By iterating we get
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\[ \phi(2^n K) \subseteq \phi(K) \text{ for any } n \in \mathbb{N}. \] As \( K \) is absorbing (see Theorem 2.1.10-5), we have \( X = \bigcup_{n=1}^{\infty} 2^n K \). Thus

\[ X/M = \phi(X) = \bigcup_{n=1}^{\infty} \phi(2^n K) \subseteq \phi(K). \]

Since \( \phi \) is continuous and the continuous image of a compact set is compact, we get that \( \phi(K) \) is compact. Thus \( X/M \) is a Hausdorff t.v.s. which is compact. We claim that \( X/M \) must be of zero dimension, i.e. reduced to one point. This concludes the proof because it implies \( \dim(X) = \dim(M) < \infty \).

Let us prove the claim by contradiction. Suppose \( \dim(X/M) > 0 \) then \( X/M \) contains a subset of the form \( \mathbb{R} \bar{x} \) for some \( \bar{o} \neq \bar{x} \in X/M \). Since such a subset is closed and \( X/M \) is compact, \( \mathbb{R} \bar{x} \) is also compact which is a contradiction. \( \square \)