Polyharmonic Cubature Formulas on the Ball and on the Sphere

Polyharmonic Paradigm

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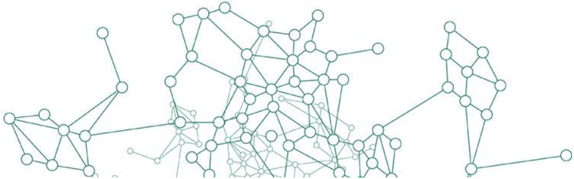

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- **G. G. Hardy, A Mathematician's Apology**

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- Spectral theory in finite dimensions – directly related to PCA and SVD - the most applied tool in analysis of Economics and Financial data

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- Completely Integrable systems, Toda and KdV - J. Moser (1974) in Toda lattices, (multivariate, Albeverio, OK, 2015)

One-dimensional reminder - Jacobi's point of view

- Remind: why are orthogonal polynomials $P_N(t)$ so valuable ?
  Gauss-Jacobi quadrature contains weight $w(t)$:

$$\int_{-1}^{1} f(t)\, w(t)\, dt = \sum_{j=1}^{N} f(t_j)\, \lambda_j \qquad \text{where } P_N(t_j) = 0.$$

Also, the Stieltjes transform

$$\int_{-1}^{1} \frac{w(t)\, dt}{t-z} \approx \frac{Q_N(z)}{P_N(z)} + O\left(\frac{1}{z^{N+1}}\right) \qquad \text{for } z \longrightarrow \infty.$$

Think about the **multidimensional analogy** to Gauss-Jacobi rules !

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- Quadrature formulas – **Gauss** (for $w \equiv 1$), and **Gauss-Jacobi**:
  $f(t) = g(t)\, w(t)$ with the weight $w(t)$ as e.g. $w(t) = t^{\alpha}(1-t)^{\beta}$.
  Compute

  $$\int_{0}^{1} g(t)\sqrt{t}\, dt$$

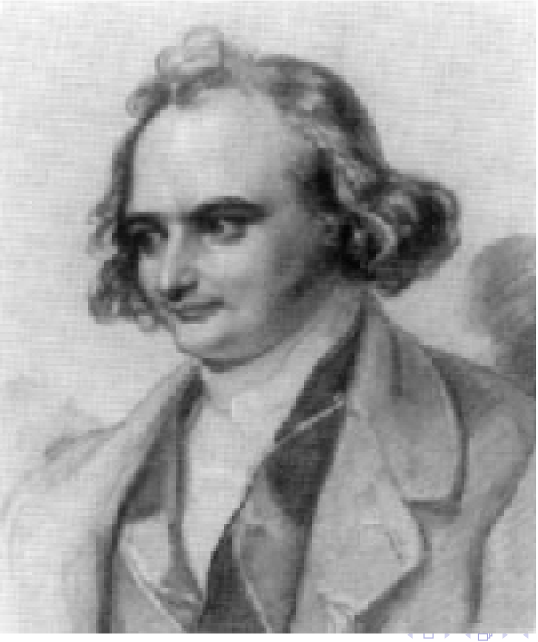
  for a polynomial $g(t)$ in two ways: using Gauss, or using Gauss-Jacobi (respecting the weight $\sqrt{t}$).

Think about the **multidimensional analogy** to Gauss-Jacobi rules !

Who are the classics in 1D Quadrature formulas?

Carl Friedrich Gauss:

Carl Gustav Jacob Jacobi:



The Moment Problem classics: Andrey Markov

The Moment Problem classics: Thomas Stieltjes

Charles Hermite (December 24, 1822 – January 14, 1901)

Famous contributors to Cubature formulas: Sobolev, (6 Oct. 1908 – 3 Jan. 1989)

S. L. Sobolev – 2

Cubature formula

We consider integrals on the **unit ball** $B \subset \mathbb{R}^n$:

$$\int_B f(x)\, dx.$$

Following Jacobi's **point of view**, assume that $f(x)$ has representation

$$f(x) = P(x)\, w(x)$$

with $P(x)$ – a polynomial; $w(x)$ – a "weight function" of a limited smoothness (or, singularity) at $x = 0$.

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- How to proceed?
- We need a new point of view on the multivariate polynomials.

A remarkable representation of multivariate polynomials

- Let $\{ Y_{k,\ell}(x) : \ell = 1, 2, ..., a_k \}$ be an orthonormal basis of the set of homogeneous (order k) harmonic polynomials - the spherical harmonics; hence

$$Y_{k,\ell}(r\theta) = r^k Y_{k,\ell}(\theta) \qquad \text{for } r = |x|, \theta \in \mathbb{S}^{n-1}.$$

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- Let $P(x)$ be a multivariate polynomial satisfying $\Delta^N P(x) = 0$. Then the following **remarkable Almansi** representation holds

$$P(x) = \sum_{k,\ell} p_{k,\ell}(r^2) Y_{k,\ell}(x) \qquad r = |x|$$

where $p_{k,\ell}$ is a $1D$ polynomial of degree $\leq N - 1$ (see about the Gauss-Almansi S. Sobolev's book on Cubature formulas).

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- Hence, the **polyharmonic degree** N (in Δ^N) is a generalization for the one-dimensional degree N of the polynomials.
- This is a fundamental point of the so-called **Polyharmonic Paradigm** (see the monograph O.K., Multivariate Polysplines, Academic press, 2001)

Two-dimensional case - simpler

For simplicity, consider the case $n = 2$, the plane $\mathbb{R}^2$ where we have $a_0 = 1$ and $a_k = 2$ for $k \geq 1$, namely,

$$Y_{(0,1)}(\varphi) = 1/\sqrt{2\pi}$$
$$Y_{(k,1)}(\varphi) = \frac{1}{\sqrt{\pi}} \cos k\varphi \quad \text{and} \quad Y_{(k,2)}(\varphi) = \frac{1}{\sqrt{\pi}} \sin k\varphi.$$

for integers $k \geq 1$.
We have the Fourier expansion of the weight function $w(x)$:

$$w(x) = \sum_{k,\ell} w_{k,\ell}(r)\, Y_{k,\ell}(\theta) \qquad \text{where}$$
$$w_{k,\ell}(r) = \int_{\mathbb{S}^{n-1}} w(r\theta)\, Y_{k,\ell}(\theta)\, d\sigma_\theta$$

The integral as infinite sum of 1-dim integrals

$$\int_B P(x)\, w(x)\, dx = \sum_{k,\ell} \int_0^1 p_{k,\ell}\left(r^2\right) r^k w_{k,\ell}(r)\, r^{n-1} dr \qquad \text{with } \rho = r^2,$$

$$= \sum_{k,\ell} \int_0^1 p_{k,\ell}(\rho)\, \widetilde{w}_{k,\ell}(\rho)\, d\rho;$$

- **Now what if (a crucial assumption !!!)**

$$d\mu_{k,\ell} \geq 0 \quad ???$$

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- Hence, for every (k,ℓ) and $N \geq 1$, apply $N-$point Gauss-Jacobi quadrature:

$$\int_0^1 p_{k,\ell}(\rho)\, \widetilde{w}_{k,\ell}(\rho)\, d\rho \approx \sum_{j=1}^{N} p_{k,\ell}\left(t_{j;k,\ell}\right) \lambda_{j;k,\ell}$$

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The cubature formula defined:

Now, let $g(x)$ be a continuous function with Fourier expansion

$$g(x) = \sum_{k,\ell} g_{k,\ell}(r)\, Y_{k,\ell}(\theta)$$

The integral becomes

$$\begin{aligned}
\int_B g(x)\, w(x)\, dx &= \sum_{k,\ell} \int_0^1 g_{k,\ell}(r)\, w_{k,\ell}(r)\, r^{n-1} dr \\
&= \frac{1}{2} \sum_{k,\ell} \int_0^1 g_{k,\ell}(\sqrt{\rho})\, \rho^{-\frac{k}{2}} \rho^{\frac{k}{2}} w_{k,\ell}(\sqrt{\rho})\, \rho^{\frac{n-2}{2}} d\rho \\
&\approx \frac{1}{2} \sum_{k,\ell} \sum_{j=1}^{N} g_{k,\ell}\left(\sqrt{t_{j;k,\ell}}\right) \rho_{j;k,\ell}^{-\frac{k}{2}} \times \lambda_{j;k,\ell} \\
&=: C(g)
\end{aligned}$$

The miracle - Chebyshev inequality applied

The last integrals are approximated as:

$$\int_0^1 g_{k,\ell}\left(\sqrt{\rho}\right)\rho^{-\frac{k}{2}}\rho^{\frac{k}{2}}w_{k,\ell}\left(\sqrt{\rho}\right)\rho^{\frac{n-2}{2}}d\rho \approx \sum_{j=1}^{N} g_{k,\ell}\left(\sqrt{t_{j;k,\ell}}\right)\cdot t_{j;k,\ell}^{-\frac{k}{2}}\cdot\lambda_{j;k,\ell}$$

Important to see, when does the following hold (?)

$$\sum_{k,\ell}\sum_{j=1}^{N} g_{k,\ell}\left(t_{j;k,\ell}\right)\cdot t_{j;k,\ell}^{-\frac{k}{2}}\cdot\lambda_{j;k,\ell} < \infty.$$

The proof: application of the famous Chebyshev inequality: Let $F\left(r\right)$ satisfy

$$F^{(2N)}\left(r\right) \geq 0.$$

THEOREM. (Chebysev-Markov-Stieltjes) The Gauss-Jacobi quadrature for $w\left(t\right)$ satisfies

$$\sum_{j=1}^{N} F\left(t_j\right)\lambda_j \leq \int_0^1 F\left(t\right)w\left(t\right)dt.$$

Chebyshev inequality

From above we obtain

$$\left| \sum_{j=1}^{N} g_{k,\ell}\left(t_{j;k,\ell}\right) \cdot t_{j;k,\ell}^{-\frac{k}{2}} \cdot \lambda_{j;k,\ell} \right| \leq C \|g\|_{\sup} \sum_{j=1}^{N} t_{j;k,\ell}^{-\frac{k}{2}} \cdot \lambda_{j;k,\ell}$$

$$\leq C \|g\|_{\sup} \int \rho^{-\frac{k}{2}} \rho^{-\frac{k}{2}} w_{k,\ell}\left(\sqrt{\rho}\right) \rho^{\frac{n-2}{2}} d\rho$$

$$= C \|g\|_{\sup} \int w_{k,\ell}\left(\sqrt{\rho}\right) \rho^{\frac{n-2}{2}} d\rho$$

If we impose the condition

$$\|w\| := \sum_{k,\ell} \int w_{k,\ell}\left(\sqrt{\rho}\right) \rho^{\frac{n-2}{2}} d\rho < \infty$$

then the sum is bounded from above.

Remember Hardy's idea for a mathematical proof

- The above application of the Chebyshev inequality corresponds to the idea of proof of Hardy - clear cut constellation.

Final approximation of the Fourier coefficients

To finish the Cubature formula, approximate the coefficients $g_{k,\ell}(r)$.
In $\mathbb{R}^2$ we have have

$$g_{k,1}(r) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} g\left(re^{i\varphi}\right) \cos k\varphi d\varphi$$

$$g_{k,2}(r) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} g\left(re^{i\varphi}\right) \sin k\varphi d\varphi$$

Hence, for integers $M \geq 1$, the approximation is just the **trapezoidal rule**:

$$f_{(k,\ell)}^{(M)}(r) := \frac{2\pi}{M} \sum_{s=1}^{M} f\left(re^{i\frac{2\pi s}{M}}\right) Y_{(k,\ell)}\left(\frac{2\pi s}{M}\right)$$

The final Cubature formula is:

$$\int_B g(x)\, w(x)\, dx \approx$$

$$\approx \frac{\pi}{M} \sum_{k=0}^{K} \sum_{\ell=1}^{a_k} \sum_{j=1}^{N} \sum_{s=1}^{M} \lambda_{j,(k,\ell)} \cdot t_{j,(k,\ell)}^{-\frac{k}{2}} \cdot Y_{(k,\ell)}\left(\frac{2\pi s}{M}\right) \cdot g\left(\sqrt{t_{j,(k,\ell)}}\, e^{i\frac{2\pi s}{M}}\right)$$

Nice properties of the Cubature formula – stability estimate

The coefficients satisfy the stability estimate

$$\left| \frac{\pi}{M} \sum_{k=0}^{K} \sum_{\ell=1}^{a_k} \sum_{j=1}^{N} \sum_{s=1}^{M} \lambda_{j,(k,\ell)} \cdot t_{j,(k,\ell)}^{-\frac{k}{2}} \cdot Y_{(k,\ell)} \left(\frac{2\pi s}{M} \right) \right| \leq C_1 \| w \| .$$

By a theorem of Polya and others, we have a stable Cubature formula.
Due to the above, we have **all nice Error estimates** for these Cubature formula.
Details are available in **arxiv**: **http://arxiv.org/abs/1509.00283**

The new orthogonal polynomials model

We generalize the one-dimensional M. Stone model for self-adjoint operators with simple spectrum, in a separable Hilbert space; see Theorem 4.2.3 in N. Akhiezer, The classical moment problem, 1965.
Here we provide the following generalization of the model: Assume that the weight function w is **pseudo-definite**, i.e.

$$w_{k,\ell}\left(r\right)>0 \quad \text{or} \quad w_{k,\ell}\left(r\right)<0 \qquad \text{for } 0\leq r\leq 1.$$

We define the space of the model:

$$L_2'\left(w\left(x\right)dx\right):=\bigoplus\nolimits_{k,\ell}L_2\left(d\mu_{k,\ell}\right)$$

where the functions $f\left(x\right)$ are represented by means of the Almansi formula. The operator is

$$Af\left(x\right)=\left|x\right|^2 f\left(x\right).$$

The basis of this space are the multivariate "orthogonal" polynomials

$$\left\{P_j^{k,\ell}\left(r^2\right)r^kY_{k,\ell}\left(\theta\right)\right\}_{j;k,\ell}$$

and they generate all polynomials by means of the Almansi formula. The

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