SOME MOMENT PROBLEMS FROM ONE TO INFINITE DIMENSIONS

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1. MOTIVATIONS, AND SOME MOMENT PROBLEMS IN Q.F.

2. SOME RELATIONS TO INFIN. DIM. S- MOMENT PROBLEM

1. MOTIVATIONS, AND SOME MOMENT PROBLEMS IN Q.F.

Ex. 1 (Non relativistic Q.M., 1 particle in $\mathbb{R}^s$)

$$\langle f(\cdot)\rangle = \text{"} Z^{-1} \int_\Gamma e^{\frac{i}{\hbar} S(\gamma)} f(\gamma)\, d\gamma \text{"}$$

Path space

path: $\gamma : [0,t] \to \mathbb{R}^s$

E.g. $f(\gamma) = \prod_{i=1}^{n} \gamma(\tau_i)$

$S(\gamma) := \int_0^t |\dot\gamma(\tau)|^2 d\tau - \int_0^t V(\gamma(\tau))d\tau$

Rem: $\langle \cdot \rangle$ $\mathbb{C}$ - functional

$\hbar \downarrow 0$: "classical limit"

Ex. 2 (Scalar) Q.F.

$$\Gamma = \text{Maps}\,(\underbrace{\mathbb{R}\times\mathbb{R}^s}_{\mathbb{R}^d} \to \mathbb{R})$$

$$\gamma = \gamma(t, \vec{x})$$

$$S(\gamma) := \int_{\mathbb{R}^d} \left[|\dot\gamma(x)|^2 - |\nabla_{\vec{x}}\gamma(x)|^2 \right] dx - \int_{\mathbb{R}^d} V(\gamma(x))\, dx$$

E.g. $f(\gamma) = \prod_{i=1}^{n} \gamma(x_i)$

$f \to \langle f \rangle$ "complex functional"

$\hbar \downarrow 0$: "classical limit"

Ex. 3 "Euclidean" versions of Ex ~~1~~, 2. [useful, many interesting applications!]:

: $S(\gamma) \rightsquigarrow S_E(\gamma)$: defined as $S(\gamma)$ but with $\int|\dot\gamma|^2 \rightsquigarrow -\int|\dot\gamma|^2$

Then $\langle f(\cdot) \rangle =$ "$\underbrace{Z^{-1}\int e^{-S_E(\gamma)}}_{\mu(d\gamma)} \underline{f(\gamma)\,d\gamma}$"

($\hbar = 1$)

If f polyn. : $\langle f \rangle$ called
"Schwinger functions"

Rem 1 "$Z^{-1} e^{-S_E(\gamma)} d\gamma$"
well def. Gaussian meas.
$\mu_0(d\gamma) := N\left(0; (-\Delta + m^2)^{-1}\right)$
on $\mathcal{S}'(\mathbb{R}^d)$, if $V = V_0(\gamma(x))$
$= m^2 \gamma(x)^2$
"Nelson's free field measure"

Rem 2 If $V = V_0 + \underbrace{\overbrace{\lambda\,(\text{higher order poly.})}^{V_1}}_{*}$

* ill defined, since γ singular

Physicists heuristic exp. of $\langle f \rangle$ in powers of λ and renormalization term by term, no summability results for $d \geq 4$

For $d = 1, 2, (3)$ rigorous results
("constructive QF theory"):

V_1 in $S_E \rightsquigarrow V_{1,\varepsilon,\Lambda}$, with

$$\int \gamma(x)^n dx \rightsquigarrow \int_\Lambda :\gamma_\varepsilon(x)^n: dx$$

(: : needed for $d \geq 2$)

$\Lambda \subset\subset \mathbb{R}^d$

$\varepsilon > 0$

Call $\mu_{\varepsilon,\Lambda}$ meas. obtained from expression for μ by these replacements:

$\mu_{\varepsilon,\Lambda} \longrightarrow \mu$ weakly in $\mathcal{S}'(\mathbb{R}^d)$

Rem: Moment problem: Zessin ....
Kondratiev, Kuna, Lytvynov

μ unique for $d = 1$

$d \leq 3$: Borel summable expansion in λ for $\langle f \rangle$

Rem: $\int_\Lambda :\gamma_\varepsilon(x)^n: dx$ called "Wick powers of γ_ε in Λ"

For $\varepsilon > 0$ def. as n-th deriv. at $\alpha = 0$ of

$$\int_\Lambda :\exp(\alpha\gamma_\varepsilon(x)): dx$$

$$:= \left(\int_\Lambda \exp(\alpha\gamma_\varepsilon(x))\, dx\right) / E_{\mu_0}(\cdot)$$

Well def. in $L^2(\mu_0)$ for $\varepsilon \downarrow 0$ if $d = 2$. Call

$\int_\Lambda :\gamma^n:(x)\, dx$ the limit for $\varepsilon \downarrow 0$:

"n-th Wick power of γ in Λ"

$\langle \varphi, :\gamma^n: \rangle$ well def. $\forall\, \varphi \in \mathcal{S}(\mathbb{R}^2)$

Rem: In relativistic case $\exists$ analogue "n-th Wick power" as "operator valued distrib. in Fock space for free field", all d!

Interesting connection with Moment Problem

n = 1 : field oper. : essent. self-adj. on natural dom.
(as seen via Nelson's anal. vectors)

n = 2 : also (via Nussbaum quasi-analytic vectors: Sachok ...)

n = 3 : densely def, real, symmetric
but open wether
es. s.a.

For d = 2, "fixed time" es. s.a.
Ferrario, Yoshida, A.
Choice of test functions plays a role: detailed results by
S. Rabsztyn '89 (and A. '68)

$\langle 4, :\gamma^3: \rangle$ expressed by creation/annih. opts According to choice of
$\mathcal{H}$: defect indices (0,0), (1,1), (3,3)

Relation to study of diff. opts in 1-d
e.g. Kostyuchenko, Mirzoev '99 ...

2. SOME RELATIONS TO INFIN. DIM. S-MOMENT PROBLEM

Yesterday's lect. by Infusino:
S-Moment Probl. on $\mathcal{D}'(\mathbb{R}^d)$

$\mathbb{R}^N$: $\ell : \mathbb{R}[X_1, \dots, X_N] \to \mathbb{R}$ linear

$\ell(1) = 1$

$S := \{x \in \mathbb{R}^N \mid (f_1 \wedge \dots \wedge f_m)(x) \geq 0\}$

$\in \mathbb{R}[X_1, \dots, X_N]$

m, f_i given

Mom. Probl.: Find $\mu \in \mathcal{M}_1(\mathbb{R}^N)$

s.t. $\int_{\mathbb{R}^N} p(y)\mu(dy) = \ell(p(\vec{X}))$,

$\vec{X} = (X_1, \dots, X_N)$, $\forall p \in \mathbb{R}[X_1, \dots, X_N]$

Schmüdgen '91 : for S compact: $\exists \mu$
necess. & suffic.

$\forall (i_1, \dots, i_n) \in \{0,1\}^n$, $\forall g \in \mathbb{R}[X_1, \dots, X_N]$

$$\ell\Big(g(\vec{X})^2 \prod_{i=1}^{m} f_i(\vec{X})^{j_i}\Big) \geq 0$$

In this case $\mu \overbrace{\{x \mid f_1 \wedge \dots \wedge f_m(x) \geq 0\}}^{S}$
$= 1$

Vast generalizations: e.g.
Kuhlmann, Marshall, Schwartz...

Yesterday: $\mathbb{R}^N \rightsquigarrow \mathcal{D}'(\mathbb{R}^d)$

$S \rightsquigarrow$ semi-algebr. subset of gener. fcts in $\mathcal{D}'(\mathbb{R}^d)$ given by polynomial constraints

Suffic.: positivity of Riesz functional assoc. with ℓ on quadratic modules

E.g. Quasi analytic criterium...
Infusino, Kuna, Rota; Ghasemi, Kuna, Infusino, Marshall ....

There should be relations with QF...

Simpler: Wiener space (or space for Ornstein-Uhlenbeck ...):

$$(-\Delta + m^2)^{-1} \rightsquigarrow \left(-\frac{d^2}{dt^2} + m^2\right)^{-1}$$

covar. of O-U process, parameter m

Paper by Frederik Herzberg + A.

Insted of $\mathcal{D}'(\mathbb{R}^d)$, $d = 1$:

$C_{(0)}([0,1];\mathbb{R})$: Wiener space

Motivation: $\mathbb{R}^N$ as path space of N-time random walk

$(X_1(\omega), \dots, X_N(\omega)) := \vec{X}(\omega)$,

$\omega \in (\Omega, \mathcal{F}_N, (\mathcal{F}_i)_{i=1,\dots,N}), P)$

$$\ell(\mathbf{p}(\vec{X})) = \int_\Omega p(\vec{X}(\omega)) P(d\omega)$$

$$P\Big(\underbrace{\bigcap_{i=1}^{m} \{f_i(\vec{X}(\omega)) \geq 0\}}_{S(\omega)\text{ set}}\Big) = 1$$

Notation:

$$\Pi_{\mathbb{Q}}(\omega) := \{ Y(\omega) \mid Y(\omega) = \prod_{j=1}^{m} X_{q_j}^{i_j}(\omega),$$
$$i_j \in \mathbb{N},\ q_j \in \mathbb{Q} \cap (0,1] \}$$

$\langle \Pi_Q(\cdot) \rangle$ generated vector space

$\Pi_{\mathbb{R}}(\omega)$ as above with
$\mathbb{Q} \cap (0,1] \rightsquigarrow (0,1]$

Let b : stand. Brown. motion on $[0,1]$

<u>Theor</u>: Let $f_1(\vec{b}), \dots, f_m(\vec{b}) \in \langle \Pi_{\mathbb{Q}} \rangle$

$\vec{b}$ vector of b taken at rational times

Assume $\ell : \mathbb{R}[\vec{b}] \to \mathbb{R}$, $\ell(1) = 1$ and

$$\ell\left((g(\vec{b}))^2\right) \prod_{i=1}^{m} f_i^{k_i}(\vec{b})) \geq 0$$

$\forall\, (k_1, \dots, k_m) \in (0,1]^m$, $g \in \langle \Pi_{\mathbb{Q}} \rangle$

Assume in addition $\exists c > 0$ s.t.

$$n \max_{k<n} \left| \ell\left(g(\vec{b})^2\right) \cdot \left(\left|b_{\frac{k+1}{n}} - b_{\frac{k}{n}}\right|^2 - c\, \ell\left(g(\vec{b})^2\right)\right)\right| \underset{n\to\infty}{\longrightarrow} 0$$

("contin. cond. on ℓ")

Then $\exists\, \tilde{b}_t,\ t \in \mathbb{R} \cap (0,1]$ B.M. s.t.

1) $f_i\left((c\,\tilde{b}_t)_{t\in\mathbb{R}\cap(0,1]} \geq 0\right) \geq 0$ a.s., $\forall i \in \{1, \dots m\}$

2) $\ell(p(\vec{b})) = E(p(c\tilde{b}))$ $\forall p(\vec{b}) \in \langle \pi_{\mathbb{Q}} \rangle$

Moreover $\exists$ L^1-cont. ext. of ℓ to $\langle \pi_{\mathbb{R}} \rangle$.

Proof: NSA (Brown. motion as infinitesimal steps random walk; Loeb measure)

"transfer" from finite-dim. ...