The operator theoretic moment problem

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What is it and why should I care?

Classical multivariate moment problem

- Dual problem to classification of positive polynomials
- Let $K \subseteq \mathbb{R}^n$ be closed.

Moment problem

Let $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ be linear, $L(1) = 1$. Does there exist a probability measure μ with $\operatorname{supp} \mu \subseteq K$ such that for all $f \in \mathbb{R}[\underline{x}]$:

$$L(f) = \int f(\underline{a}) \, \mathrm{d}\mu(\underline{a})?$$

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What do I need it for?

- Applications in quantum physics
  - quantum chemistry: ground state electronic energy of atoms
  - quantum theory: upper bounds for violation of Bell inequalities
  - quantum information: multi prover games/quantum correlation
  - Genral: non-commutative probability theory
- Application in systems control
  - Systematic strategy to compute stabilizing feedback for linear closed loop systems

NC polynomials

- $\underline{X} = (X_1, \ldots, X_n)$ non-commuting/free variables
- $\mathbb{R}\langle \underline{X} \rangle$ unital associative free algebra generated by $\underline{X}$
- Elements $f \in \mathbb{R}\langle \underline{X} \rangle$ are NC polynomials
- Involution $^* : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}\langle \underline{X} \rangle$ s.t.
  - X_i self-adjoint
  - * is identity on $\mathbb{R}$
- Evaluation in symmetric matrices or self-adjoint operators
  - $f(\underline{A}) = f_1 \mathbf{1} + f_{X_1} A_1 + f_{X_2} A_2 + \ldots + f_{X_1^2 X_3 X_2^3} A_1^2 A_3 A_2^3 + \ldots$

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Moment problems for linear forms
$L : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}$

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- Consider 2 types of positivity of NC polynomials

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  - Positivity by eigenvalue
    - $f \in \mathbb{R}\langle \underline{X} \rangle$ is positive semidefinite if

      $$f(\underline{A}) \succeq 0 \text{ for all tuples } \underline{A} \text{ of symmetric matrices of any size.}$$

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    - Can be extended to self-adjoint bounded operators
  - Positivity by trace
    - $f \in \mathbb{R}\langle \underline{X} \rangle$ is trace-positive if

      $$\mathrm{Tr}(f(\underline{A})) \geq 0 \text{ for all tuples } \underline{A} \text{ of symmetric matrices of any size.}$$
    - Can be extended to finite von Neumann algebras

Operator theoretic moment problems

CWI

NC moment problem

For which linear form $L : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}$ exists a finite dimensional Hilbert space H, a unit vector $\phi \in H$ and a $*$-representation π on $B(H)$ such that for all $f \in \mathbb{R}\langle \underline{X} \rangle$:

$$L(f) = \langle \pi(f)\phi, \phi \rangle ?$$

Tracial moment problem

For which linear form $L : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}$ exists some $s \in \mathbb{N}$ and a probability measure μ with $\operatorname{supp} \mu \subseteq (\mathbb{S}\mathbb{R}^{s \times s})^n$ such that for all $f \in \mathbb{R}\langle \underline{X} \rangle$:

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Can also formulate
K-moment problems

Hankel matrices

- Associate to $L : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}$ the sesquilinear form

$$B_L : \mathbb{R}\langle \underline{X} \rangle \times \mathbb{R}\langle \underline{X} \rangle, (f, g) \mapsto L(f^* g).$$

- The representing matrix for B_L is its Hankel matrix

Definition

- The Hankel matrix $M(L)$, indexed by $u, v \in \langle \underline{X} \rangle$, is given by

$$M(L)_{u,v} := L(u^* v).$$

- The truncated Hankel matrix $M_k(L)$ of degree k is the submatrix of $M(L)$ indexed by $u, v \in \langle \underline{X} \rangle_k$.

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For K-moment problem use also localizing Hankel matrices

One Hankel matrix

Example

Consider $\mathbb{R}\langle X, Y\rangle$ with basis $(1, X, Y, X^2, XY, YX, \dots)$

$$M(L) = \begin{bmatrix} L(1) & L(X) & L(Y) & L(X^2) & L(XY) & \dots \\ L(X) & L(X^2) & L(XY) & L(X^3) & L(X^2Y) & \dots \\ L(Y) & L(YX) & L(Y^2) & L(YX^2) & L(YXY) & \dots \\ L(X^2) & L(X^3) & L(X^2Y) & L(X^4) & L(X^3Y) & \dots \\ L(YX) & L(YX^2) & L(YXY) & L(YX^3) & L(YX^2Y) & \dots \\ L(XY) & L(XYX) & L(XY^2) & L(XYX^2) & L(XYXY) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Truncated NC moment problem

Proposition (Helton, Klep, McCullough)

$L : \mathbb{R}\langle \underline{X} \rangle_{2d} \to \mathbb{R}$ has a finite dimensional moment representation iff

1. $M_d(L) \succeq 0$
2. for some $k > d$ exists a flat Hankel matrix extension M_k of $M_d(L)$, i.e., $\operatorname{rank} M_k = \operatorname{rank} M_{k-1}$.

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Can also formulate
K-moment problem version

Full NC moment problem

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3. there exists a $C \in \mathbb{R}_{\geq 0}$ such that $M[C - X_i^2, L] \succeq 0$ for all $i \in [n]$.

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- Allow infinite dimensional Hilbert spaces:

Theorem (Pironio, Navascues, Acin)

$L : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}$ has a moment representation if and only if

1. $M(L) \succeq 0$
2. there exists a $C \in \mathbb{R}_{\geq 0}$ such that $M[C - \sum_i X_i^2, L] \succeq 0$.

The tracial moment problem

- Additional constraint on $L : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}$

Tracial condition

$$L(fg) = L(gf) \text{ for all } f, g \in \mathbb{R}\langle \underline{X} \rangle$$

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Proposition (B.)

A tracial $L : \mathbb{R}\langle \underline{X} \rangle_{2d} \to \mathbb{R}$ has a finite dimensional tracial moment representation iff

1. $M_d(L) \succeq 0$
2. for some $k > d$ exists a flat (tracial) Hankel matrix extension M_k of $M_d(L)$.

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For K-moment problem add psd localizing Hankel matrices

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Proposition (Klep, Schweighofer;B.)

A tracial $L : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}$ has a tracial moment representation (using a von Neumann algebra) if and only if

1. $M(L) \succeq 0$
2. there exists a $C \in \mathbb{R}_{\geq 0}$ such that $M[C - \sum_i X_i^2, L] \succeq 0$.

Application in NC Polynomial optimization

- $p \in \mathbb{R}\langle \underline{X} \rangle$ nc polynomial

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$$\boxed{p_{sos} = \max\{\lambda \mid p - \lambda \text{ sos}\}}$$

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$$\boxed{p_{ds} = \min\{L(p) \mid L \in \mathbb{R}\langle \underline{X} \rangle^{\vee}, M(L) \succeq 0\}}$$

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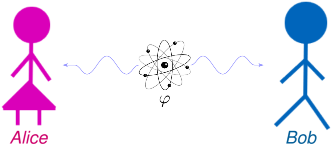
$$\boxed{p_{ds} = \min\{L(p) \mid L \in \mathbb{R}\langle \underline{X} \rangle^{\vee}, M(L) \succeq 0\}}$$

- If optimizing L in p_{ds} has a moment representation then

$$p_{min} \leq p_{sos} \leq p_{ds} = L(p) \leq p_{min}$$

- Moment representation implies exactness of relaxation

Application: Quantum Correlations



- Two separated systems: A and and B
- Measurements:
  described by operators E_i performed on a joint quantum state φ
- Local measurements:
  Alice's operators E_i commute with bob's operators E_j
- Correlations between A and B:
  Joint probabilities $P(i,j) = \langle \varphi, E_i E_j \varphi \rangle$

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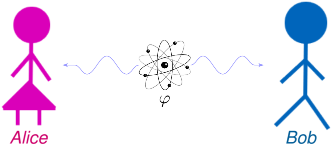

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  Joint probabilities $P(i, j) = \langle \varphi, E_i E_j \varphi \rangle$
- Violation of Bell inqualities
  - Linear combination of (joint) probabilities
  - Get inequalities by considering classical random variables
  - Want to find violations using quantum setup

Application: Quantum Correlations

- Violation of Bell inequalities
- Given linear relation $\sum_{i,j} c_{i,j} P(i,j)$

$$
\begin{aligned}
\max_{(E,\varphi)} \quad & \left\langle \varphi, \sum_{i,j} c_{ij} E_i E_j \varphi \right\rangle \\
\text{s.t.} \quad & \|\varphi\| = 1 \\
& E_i E_j = \delta_{ij} \text{ for } i,j \in M_k \\
& \sum_{i \in M_k} E_i = 1 \\
& [E_i, E_j] = 0 \text{ for } i \in A, j \in B
\end{aligned}
$$

$\leftarrow \langle \varphi, p(\underline{E})\varphi \rangle$

$\leftarrow \|\varphi\| = 1$

] measurement

] A/B separated

Application: Quantum Correlations

CWI

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- Pal, Vertesi run sos-relaxation on 241 Bell inequalities
- prove exactness for about 220 of them

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- Quantum field model of measurements leads to a version with tracial moments

Conclusion

- Operator theoretic moment problems
  - eigenvalue/psd version: $L(p) = \langle \varphi\, p(\underline{A}), \varphi \rangle$
  - trace version: $L(p) = \mathrm{Tr}(p(\underline{A}))$
- Generalizes the classical moment problem
  - A lot of statements remain true...
  - ... if one allows infinite dimensional Hilbert spaces

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Challenge: How can we distinguish between finite and infinite dimensions apart from checking for flatness?