Saturation of preorderings in $\mathbb{R}[\underline{X}]$

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Abstract

Let T_S be a preordering of the polynomial ring $\mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \dots, X_n]$ finitely generated by the subset $S = \{g_1, \dots, g_s\}$ of $\mathbb{R}[\underline{X}]$ and K_S the basic closed semi-algebraic set defined by S, i.e. the subset of $\mathbb{R}^n$ where $g_1, \dots, g_s$ are non-negative. We want to examine under which conditions T_S is saturated, that is, when does $f \geq 0$ on K_S implies that $f \in T_S$. Due to a theorem of Scheiderer, if the dimension of the semi-algebraic set K_S is equal or higher than 3, then T_S is never saturated. In lower dimensions there are examples which show that T_S can and cannot be saturated, depending on the properties of S. In the case $n = 1$ if S is the natural description of the closed semi-algebraic set K_S in $\mathbb{R}$, then the preordering T_S is saturated. For $n = 2$ there is no such a result, nevertheless we will see, that if K_S contains a 2-dimensional affine cone, then T_S is not saturated.