Topological Vector Spaces

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Chapter 1

Preliminaries

1.1 Topological spaces

1.1.1 The notion of topological space

The topology on a set X is usually defined by specifying its open subsets of X. However, in dealing with topological vector spaces, it is often more convenient to define a topology by specifying what the neighbourhoods of each point are.

Definition 1.1.1. *A* topology τ *on a set* X *is a family of subsets of* X *which satisfies the following conditions:*
(O1) *the empty set* $\emptyset$ *and the whole* X *are both in* τ
(O2) τ *is closed under finite intersections*
(O3) τ *is closed under arbitrary unions*
The pair (X, τ) *is called a* topological space*.*

The sets $O \in \tau$ are called *open sets* of X and their complements $C = X \setminus O$ are called *closed sets* of X. A subset of X may be neither closed nor open, either closed or open, or both. A set that is both closed and open is called a *clopen set*.

Definition 1.1.2. *Let* (X, τ) *be a topological space.*

- *A subfamily* $\mathcal{B}$ *of* τ *is called a* basis *if every open set can be written as a union of sets in* $\mathcal{B}$*.*
- *A subfamily* $\mathcal{X}$ *of* τ *is called a* subbasis *if the finite intersections of its sets form a basis, i.e. every open set can be written as a union of finite intersections of sets in* $\mathcal{X}$*.*

Therefore, a topology τ on X is completely determined by a basis or a subbasis.

Example 1.1.3. *Let $\mathcal{S}$ be the collection of all semi-infinite intervals of the real line of the forms $(-\infty, a)$ and $(a, +\infty)$, where $a \in \mathbb{R}$. $\mathcal{S}$ is not a base for any topology on $\mathbb{R}$. To show this, suppose it were. Then, for example, $(-\infty, 1)$ and $(0, \infty)$ would be in the topology generated by S, being unions of a single base element, and so their intersection $(0, 1)$ would be by the axiom (O2) of topology. But $(0, 1)$ clearly cannot be written as a union of elements in $\mathcal{S}$.*

Proposition 1.1.4. *Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. $\mathcal{B}$ is a basis for a topology τ on X iff the following hold:*

1. *$\mathcal{B}$ covers X, i.e. $\forall\, x \in X$, $\exists\, B \in \mathcal{B}$ s.t. $x \in B$.*
2. *If $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then $\exists\, B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subseteq B_1 \cap B_2$.*

Proof. (Sheet 1, Exercise 1) $\square$

Definition 1.1.5. *Let (X, τ) be a topological space and $x \in X$. A subset U of X is called a* neighbourhood *of x if it contains an open set containing the point x, i.e. $\exists\, O \in \tau$ s.t. $x \in O \subseteq U$. The family of all neighbourhoods of a point $x \in X$ is denoted by $\mathcal{F}(x)$.*

In order to define a topology on a set by the family of neighbourhoods of each of its points, it is convenient to introduce the notion of filter. Note that the notion of filter is given on a set which does not need to carry any other structure. Thus this notion is perfectly independent of the topology.

Definition 1.1.6. *A* filter *on a set X is a family $\mathcal{F}$ of subsets of X which fulfills the following conditions:*

(F1) *the empty set $\emptyset$ does not belong to $\mathcal{F}$*
(F2) *$\mathcal{F}$ is closed under finite intersections*
(F3) *any subset of X containing a set in $\mathcal{F}$ belongs to $\mathcal{F}$*

Definition 1.1.7. *A family $\mathcal{B}$ of subsets of X is called a* basis of a filter *$\mathcal{F}$ if*

1. *$\mathcal{B} \subseteq \mathcal{F}$*
2. *$\forall A \in \mathcal{F}$, $\exists\, B \in \mathcal{B}$ s.t. $B \subseteq A$*

Examples 1.1.8.

a) *The family $\mathcal{G}$ of all subsets of a set X containing a fixed non-empty subset A is a filter and $\mathcal{B} = \{A\}$ is its base. $\mathcal{G}$ is called the* principle filter *generated by A.*
b) *Given a topological space X and $x \in X$, the family $\mathcal{F}(x)$ is a filter.*
c) *Let $S := \{x_n\}_{n \in \mathbb{N}}$ be a sequence of points in a set X. Then the family $\mathcal{F} := \{A \subset X : |S \setminus A| < \infty\}$ is a filter and it is known as the* filter associated *to S. For each $m \in \mathbb{N}$, set $S_m := \{x_n \in S : n \geq m\}$. Then $\mathcal{B} := \{S_m : m \in \mathbb{N}\}$ is a basis for $\mathcal{F}$.*

Proof. (Sheet 1, Exercise 2). $\square$

Theorem 1.1.9. *Given a topological space X and a point $x \in X$, the filter of neighbourhoods $\mathcal{F}(x)$ satisfies the following properties.*
(N1) *For any $A \in \mathcal{F}(x)$, $x \in A$.*
(N2) *For any $A \in \mathcal{F}(x)$, $\exists\, B \in \mathcal{F}(x)$: $\forall\, y \in B$, $A \in \mathcal{F}(y)$.*
Viceversa, if for each point x in a set X we are given a filter $\mathcal{F}_x$ fulfilling the properties (N1) and (N2) then there exists a unique topology τ s.t. for each $x \in X$, $\mathcal{F}_x$ is the family of neighbourhoods of x, i.e. $\mathcal{F}_x \equiv \mathcal{F}(x)$, $\forall x \in X$.

This means that a topology on a set is uniquely determined by the family of neighbourhoods of each of its points.

Proof.
$\Rightarrow$ Let (X, τ) be a topological space, $x \in X$ and $\mathcal{F}(x)$ the filter of neighbourhoods of x. Then (N1) trivially holds by definition of neighbourhood of x. To show (N2), let us take $A \in \mathcal{F}(x)$. Since A is a neighbourhood of x, there exists $B \in \tau$ s.t. $x \in B \subseteq A$. Then clearly $B \in \mathcal{F}(x)$. Moreover, since for any $y \in B$ we have that $y \in B \subseteq A$ and B is open, we can conclude that $A \in \mathcal{F}(y)$.
$\Leftarrow$ Assume that for any $x \in X$ we have a filter $\mathcal{F}_x$ fulfilling (N1) and (N2). Let us define $\tau := \{O \subseteq X : \text{if } x \in O \text{ then } O \in \mathcal{F}_x\}$. Since each $\mathcal{F}_x$ is a filter, τ is a topology. Indeed:

- $\emptyset \in \tau$ by definition of τ. Also $X \in \tau$, because for any $x \in X$ and any $A \in \mathcal{F}_x$ we clearly have $X \supseteq A$ and so by (F3) $X \in \mathcal{F}_x$.
- By (F2) we have that τ is closed under finite intersection.
- Let U be an arbitrary union of sets $U_i \in \tau$ and let $x \in U$. Then there exists at least one i s.t. $x \in U_i$ and so $U_i \in \mathcal{F}_x$ because $U_i \in \tau$. But $U \supseteq U_i$, then by (F3) we get that $U \in \mathcal{F}_x$ and so $U \in \tau$.

It remains to show that τ on X is actually s.t. $\mathcal{F}_x \equiv \mathcal{F}(x)$, $\forall x \in X$.

- Any $U \in \mathcal{F}(x)$ is a neighbourhood of x and so there exists $O \in \tau$ s.t. $x \in O \subseteq U$. Then, by definition of τ, we have $O \in \mathcal{F}_x$ and so (F3) implies that $U \in \mathcal{F}_x$. Hence, $\mathcal{F}(x) \subseteq \mathcal{F}_x$.
- Let $U \in \mathcal{F}_x$ and set $W := \{y \in U : U \in \mathcal{F}_y\} \subseteq U$. Since $x \in U$ by (N1), we also have $x \in W$. Moreover, if $y \in W$ then by (N2) there exists $V \in \mathcal{F}_y$ s.t. $\forall z \in V$ we have $U \in \mathcal{F}_z$. This means that $z \in W$ and so $V \subseteq W$. Then $W \in \mathcal{F}_y$ by (F3). Hence, we have showed that if $y \in W$ then $W \in \mathcal{F}_y$, i.e. $W \in \tau$. Summing up, we have just constructed an open set W s.t. $x \in W \subseteq U$, i.e. $U \in \mathcal{F}(x)$, and so $\mathcal{F}_x \subseteq \mathcal{F}(x)$.

$\square$

Definition 1.1.10. *Given a topological space X, a basis $\mathcal{B}(x)$ of the filter of neighbourhoods $\mathcal{F}(x)$ of a point $x \in X$ is called a* base of neighbourhoods of x, *i.e. $\mathcal{B}(x)$ is a subcollection of $\mathcal{F}(x)$ s.t. every neighbourhood in $\mathcal{F}(x)$ contains* *one in $\mathcal{B}(x)$. The elements of $\mathcal{B}(x)$ are called* basic neighbourhoods of x. *If a base of neighbourhoods is given for each point $x \in X$, we speak of* base of neighbourhoods of X.

Example 1.1.11. *The open sets of a topological space other than the empty set always form a base of neighbourhoods.*

Theorem 1.1.12. *Given a topological space X and a point $x \in X$, a base of open neighbourhoods $\mathcal{B}(x)$ satisfies the following properties.*
(B1) *For any $U \in \mathcal{B}(x)$, $x \in U$.*
(B2) *For any $U_1, U_2 \in \mathcal{B}(x)$, $\exists U_3 \in \mathcal{B}(x)$ s.t. $U_3 \subseteq U_1 \cap U_2$.*
(B3) *If $y \in U \in \mathcal{B}(x)$, then $\exists W \in \mathcal{B}(y)$ s.t. $W \subseteq U$.*
Viceversa, if for each point x in a set X we are given a collection of subsets $\mathcal{B}_x$ fulfilling the properties (B1), (B2) and (B3) then there exists a unique topology τ s.t. for each $x \in X$, $\mathcal{B}_x$ is a base of neighbourhoods of x, i.e. $\mathcal{B}_x \equiv \mathcal{B}(x)$, $\forall x \in X$.

Proof. The proof easily follows by using Theorem 1.1.9. □

The previous theorem gives a further way of introducing a topology on a set. Indeed, starting from a base of neighbourhoods of X, we can define a topology on X by setting that a set is open iff whenever it contains a point it also contains a basic neighbourhood of the point. Thus a topology on a set X is uniquely determined by a base of neighbourhoods of each of its points.

1.1.2 Comparison of topologies

Any set X may carry several different topologies. When we deal with topological vector spaces, we will very often encounter this situation of a set, in fact a vector space, carrying several topologies (all compatible with the linear structure, in a sense that is going to be specified soon). In this case, it is convenient being able to compare topologies.

Definition 1.1.13. *Let τ, τ' be two topologies on the same set X. We say that τ is* coarser *(or* weaker*) than τ', in symbols $\tau \subseteq \tau'$, if every subset of X which is open for τ is also open for τ', or equivalently, if every neighborhood of a point in X w.r.t. τ is also a neighborhood of that same point in the topology τ'. In this case τ' is said to be* finer *(or* stronger*) than τ'.*

Denote by $\mathcal{F}(x)$ and $\mathcal{F}'(x)$ the filter of neighbourhoods of a point $x \in X$ w.r.t. τ and w.r.t. τ', respectively. Then: τ is coarser than τ' iff for any point $x \in X$ we have $\mathcal{F}(x) \subseteq \mathcal{F}'(x)$ (this means that every subset of X which belongs to $\mathcal{F}(x)$ also belongs to $\mathcal{F}'(x)$).

Two topologies τ and τ' on the same set X *coincide* when they give the same open sets or the same closed sets or the same neighbourhoods of each point; equivalently, when τ is both coarser and finer than τ'. Two basis of neighbourhoods of a set are *equivalent* when they define the same topology.

Remark 1.1.14. *Given two topologies on the same set, it may very well happen that none is finer than the other. If it is possible to establish which one is finer, then we say that the two topologies are* comparable.

Example 1.1.15.
The cofinite topology τ_c *on* $\mathbb{R}$, *i.e.* $\tau_c := \{U \subseteq \mathbb{R} : U = \emptyset \text{ or } \mathbb{R} \setminus U \text{ is finite}\}$, *and the topology* τ_i *having* $\{(-\infty, a) : a \in \mathbb{R}\}$ *as a basis are incomparable. In fact, it is easy to see that* $\tau_i = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ *as these are the unions of sets in the given basis. In particular, we have that* $\mathbb{R} - \{0\}$ *is in* τ_c *but not* τ_i. *Moreover, we have that* $(-\infty, 0)$ *is in* τ_i *but not* τ_c. *Hence,* τ_c *and* τ_i *are incomparable.*

It is always possible to construct at least two topologies on every set X by choosing the collection of open sets to be as large as possible or as small as possible:

- the *trivial topology*: every point of X has only one neighbourhood which is X itself. Equivalently, the only open subsets are $\emptyset$ and X. The only possible basis for the trivial topology is $\{X\}$.
- the *discrete topology*: given any point $x \in X$, every subset of X containing x is a neighbourhood of x. Equivalently, every subset of X is open (actually clopen). In particular, the singleton $\{x\}$ is a neighbourhood of x and actually is a basis of neighbourhoods of x. The collection of all singletons is a basis for the discrete topology.

Note that the discrete topology on a set X is finer than any other topology on X, while the trivial topology is coarser than all the others. Topologies on a set form thus a partially ordered set, having a maximal and a minimal element, respectively the discrete and the trivial topology.

A useful criterion to compare topologies on the same set is the following:

Theorem 1.1.16 (Hausdorff's criterion)**.**
For each $x \in X$, *let* $\mathcal{B}(x)$ *a base of neighbourhoods of* x *for a topology* τ *on* X *and* $\mathcal{B}'(x)$ *a base of neighbourhoods of* x *for a topology* τ' *on* X.
$\tau \subseteq \tau'$ *iff* $\forall\, x \in X, \forall\, U \in \mathcal{B}(x)\ \exists\, V \in \mathcal{B}'(x)$ *s.t.* $x \in V \subseteq U$.

The Hausdorff criterion could be paraphrased by saying that smaller neighborhoods make larger topologies. This is a very intuitive theorem, because the smaller the neighbourhoods are the easier it is for a set to contain neighbourhoods of all its points and so the more open sets there will be.

Proof.

$\Rightarrow$ Suppose $\tau \subseteq \tau'$. Fixed any point $x \in X$, let $U \in \mathcal{B}(x)$. Then, since U is a neighbourhood of x in (X, τ), there exists $O \in \tau$ s.t. $x \in O \subseteq U$. But $O \in \tau$ implies by our assumption that $O \in \tau'$, so U is also a neighbourhood of x in (X, τ'). Hence, by Definition 1.1.10 for $\mathcal{B}'(x)$, there exists $V \in \mathcal{B}'(x)$ s.t. $V \subseteq U$.

$\Leftarrow$ Conversely, let $W \in \tau$. Then for each $x \in W$, since $\mathcal{B}(x)$ is a base of neighbourhoods w.r.t. τ, there exists $U \in \mathcal{B}(x)$ such that $x \in U \subseteq W$. Hence, by assumption, there exists $V \in \mathcal{B}'(x)$ s.t. $x \in V \subseteq U \subseteq W$. Then $W \in \tau'$. $\square$

1.1.3 Reminder of some simple topological concepts

Definition 1.1.17. *Given a topological space (X, τ) and a subset S of X, the* subset or induced topology *on S is defined by $\tau_S := \{S \cap U \mid U \in \tau\}$. That is, a subset of S is open in the subset topology if and only if it is the intersection of S with an open set in (X, τ).*
Alternatively, we can define the subspace topology for a subset S of X as the coarsest topology for which the inclusion map $\iota : S \hookrightarrow X$ is continuous.

Note that (S, τ_s) is a topological space in its own.

Definition 1.1.18. *Given a collection of topological space (X_i, τ_i), where $i \in I$ (I is an index set possibly uncountable), the* product topology *on the Cartesian product $X := \prod_{i \in I} X_i$ is defined in the following way: a set U is open in X iff it is an arbitrary union of sets of the form $\prod_{i \in I} U_i$, where each $U_i \in \tau_i$ and $U_i \neq X_i$ for only finitely many i.*
Alternatively, we can define the product topology to be the coarsest topology for which all the canonical projections $p_i : X \to X_i$ are continuous.

Given a topological space X, we define:

Definition 1.1.19.

- *The* closure *of a subset $A \subseteq X$ is the smallest closed set containing A. It will be denoted by $\bar{A}$. Equivalently, $\bar{A}$ is the intersection of all closed subsets of X containing A.*
- *The* interior *of a subset $A \subseteq X$ is the largest open set contained in it. It will be denoted by $\mathring{A}$. Equivalently, $\mathring{A}$ is the union of all open subsets of X contained in A.*

Proposition 1.1.20. *Given a top. space X and $A \subseteq X$, the following hold.*

- *A point x is a* closure point *of A, i.e. $x \in \bar{A}$, if and only if each neighborhood of x has a nonempty intersection with A.*
- *A point x is an* interior point *of A, i.e. $x \in \mathring{A}$, if and only if there exists a neighborhood of x which entirely lies in A.*
- *A is closed in X iff $A = \bar{A}$.*
- *A is open in X iff $A = \mathring{A}$.*

Proof. (Sheet 2, Exercise 1)

Example 1.1.21. *Let τ be the standard euclidean topology on $\mathbb{R}$. Consider $X := (\mathbb{R}, \tau)$ and $Y := \big((0,1], \tau_Y\big)$, where τ_Y is the topology induced by τ on $(0,1]$. The closure of $(0, \frac{1}{2})$ in X is $[0, \frac{1}{2}]$, but its closure in Y is $(0, \frac{1}{2}]$.*

Definition 1.1.22. *Let A and B be two subsets of the same topological space X. A is* dense *in B if $B \subseteq \bar{A}$. In particular, A is said to be dense in X (or* everywhere dense*) if $\bar{A} = X$.*

Examples 1.1.23.

- *Standard examples of sets everywhere dense in the real line $\mathbb{R}$ (with the euclidean topology) are the set of rational numbers $\mathbb{Q}$ and the one of irrational numbers $\mathbb{R} - \mathbb{Q}$.*
- *A set X is equipped with the discrete topology if and only if the whole space X is the only dense set in itself.*

  If X has the discrete topology then every subset is equal to its own closure (because every subset is closed), so the closure of a proper subset is always proper. Conversely, if X is the only dense subset of itself, then for every proper subset A its closure $\bar{A}$ is also a proper subset of X. Let $y \in X$ be arbitrary. Then to $\overline{X \setminus \{y\}}$ is a proper subset of X and so it has to be equal to its own closure. Hence, $\{y\}$ is open. Since y is arbitrary, this means that X has the discrete topology.
- *Every non-empty subset of a set X equipped with the trivial topology is dense, and every topology for which every non-empty subset is dense must be trivial.*

  If X has the trivial topology and A is any non-empty subset of X, then the only closed subset of X containing A is X. Hence, $\bar{A} = X$, i.e. A is dense in X. Conversely, if X is endowed with a topology τ for which every non-empty subset is dense, then the only non-empty subset of X which is closed is X itself. Hence, $\emptyset$ and X are the only closed subsets of τ. This means that X has the trivial topology.

Proposition 1.1.24. *Let X be a topological space and $A \subset X$. A is dense in X if and only if every nonempty open set in X contains a point of A.*

Proof. If A is dense in X, then by definition $\bar{A} = X$. Let O be any nonempty open subset in X. Then for any $x \in O$ we have that $x \in \bar{A}$ and $O \in \mathcal{F}(x)$. Therefore, by Proposition 1.1.20, we have that $O \cap A \neq \emptyset$. Conversely, let $x \in X$. By definition of neighbourhood, for any $U \in \mathcal{F}(x)$ there exists an open subset O of X s.t. $x \in O \subseteq U$. Then $U \cap A \neq \emptyset$ since O contains a point of A by our assumption. Hence, by Proposition 1.1.20, we get $x \in \bar{A}$ and so that A is dense in X. □

Definition 1.1.25. *A topological space X is said to be separable if there exists a countable dense subset of X.*

Example 1.1.26.

- *$\mathbb{R}$ with the euclidean topology is separable.*
- *The space $\mathcal{C}([0,1])$ of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the uniform topology is separable, since by the Weirstrass approximation theorem $\overline{\mathbb{Q}[x]} = \mathcal{C}([0,1])$.*

Let us briefly consider now the notion of convergence.
First of all let us concern with filters. When do we say that a filter $\mathcal{F}$ on a topological space X converges to a point $x \in X$? Intuitively, if $\mathcal{F}$ has to converge to x, then the elements of $\mathcal{F}$, which are subsets of X, have to get somehow "smaller and smaller" about x, and the points of these subsets need to get "nearer and nearer" to x. This can be made more precise by using neighborhoods of x: we want to formally express the fact that, however small a neighborhood of x is, it should contain some subset of X belonging to the filter $\mathcal{F}$ and, consequently, all the elements of $\mathcal{F}$ which are contained in that particular one. But in view of Axiom (F3), this means that the neighborhood of x under consideration must itself belong to the filter $\mathcal{F}$, since it must contain some element of $\mathcal{F}$.

Definition 1.1.27. *Given a filter $\mathcal{F}$ in a topological space X, we say that it converges to a point $x \in X$ if every neighborhood of x belongs to $\mathcal{F}$, in other words if $\mathcal{F}$ is finer than the filter of neighborhoods of x.*

We recall now the definition of convergence of a sequence to a point and we see how it easily connects to the previous definition.

Definition 1.1.28. *Given a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ in a topological space X, we say that it converges to a point $x \in X$ if for any $U \in \mathcal{F}(x)$ there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.*

If we now consider the filter $\mathcal{F}_S$ associated to the sequence $S := \{x_n\}_{n\in\mathbb{N}}$, i.e. $\mathcal{F}_S := \{A \subset X : |S \setminus A| < \infty\}$, then it is easy to see that:

Proposition 1.1.29. *Given a sequence of points $S := \{x_n\}_{n\in\mathbb{N}}$ in a topological space X, S converges to a point $x \in X$ if and only if the associated filter $\mathcal{F}_S$ converges to x.*

Proof. Set for each $m \in \mathbb{N}$, set $S_m := \{x_n \in S : n \geq m\}$. By Definition 1.1.28, S converges to x iff $\forall U \in \mathcal{F}(x), \exists N \in \mathbb{N} : S_N \subseteq U$. As $\mathcal{B} := \{S_m : m \in \mathbb{N}\}$ is a basis for $\mathcal{F}_S$ (see Problem Sheet 1, Exercise 2 c)) , we have that $\forall U \in \mathcal{F}(x), \exists N \in \mathbb{N} : S_N \subseteq U$ is equivalent to say that $\mathcal{F}(x) \subseteq \mathcal{F}_S$. ☐

1.1.4 Mappings between topological spaces

Let (X, τ_X) and (Y, τ_Y) be two topological spaces.

Definition 1.1.30. *A map $f : X \to Y$ is* continuous *if the preimage of any open set in Y is open in X, i.e. $\forall U \in \tau_Y$, $f^{-1}(U) := \{x \in X : f(x) \in Y\} \in \tau_X$. Equivalently, given any point $x \in X$ and any $N \in \mathcal{F}(f(x))$ in Y, the preimage $f^{-1}(N) \in \mathcal{F}(x)$ in X.*

Examples 1.1.31.

- *Any constant map $f : X \to Y$ is continuous.*
  Suppose that $f(x) := y$ for all $x \in X$ and some $y \in Y$. Let $U \in \tau_Y$. If $y \in U$ then $f^{-1}(U) = X$ and if $y \notin U$ then $f^{-1}(U) = \emptyset$. Hence, in either case, $f^{-1}(U)$ is open in τ_X.
- *If $g : X \to Y$ is continuous, then the restriction of g to any subset S of X is also continuous w.r.t. the subset topology induced on S by the topology on X.*
- *Let X be a set endowed with the discrete topology, Y be a set endowed with the trivial topology and Z be any topological space. Any maps $f : X \to Z$ and $g : Z \to Y$ are continuous.*

Definition 1.1.32. *A mapping $f : X \to Y$ is* open *if the image of any open set in X is open in Y, i.e. $\forall V \in \tau_X$, $f(V) := \{f(x) : x \in X\} \in \tau_Y$. In the same way, a* closed mapping *$f : X \to Y$ sends closed sets to closed sets.*

Note that a map may be open, closed, both, or neither of them. Moreover, open and closed maps are not necessarily continuous.

Example 1.1.33. *If Y has the discrete topology (i.e. all subsets are open and closed) then every function $f : X \to Y$ is both open and closed (but not necessarily continuous). For example, if we take the standard euclidean topology on $\mathbb{R}$ and the discrete topology on $\mathbb{Z}$ then the floor function $\mathbb{R} \to \mathbb{Z}$ is open and closed, but not continuous. (Indeed, the preimage of the open set $\{0\}$ is $[0, 1) \subset \mathbb{R}$, which is not open in the standard euclidean topology).*

If a continuous map f is one-to-one, f^{-1} does not need to be continuous.

Example 1.1.34.
Let us consider $[0,1) \subset \mathbb{R}$ *and* $S^1 \subset \mathbb{R}^2$ *endowed with the subspace topologies given by the euclidean topology on* $\mathbb{R}$ *and on* $\mathbb{R}^2$*, respectively. The map*

$$\begin{array}{rcc} f : [0,1) & \to & S^1 \\ t & \mapsto & (\cos 2\pi t, \sin 2\pi t). \end{array}$$

is bijective and continuous but f^{-1} *is not continuous, since there are open subsets of* $[0,1)$ *whose image under* f *is not open in* S^1*. (For example,* $[0, \frac{1}{2})$ *is open in* $[0,1)$ *but* $f([0, \frac{1}{2}))$ *is not open in* S^1*.)*

Definition 1.1.35. *A one-to-one map* f *from* X *onto* Y *is a* homeomorphism *if and only if* f *and* f^{-1} *are both continuous. Equivalently, iff* f *and* f^{-1} *are both open (closed). If such a mapping exists,* X *and* Y *are said to be two* homeomorphic topological spaces*.*

In other words an homeomorphism is a one-to-one mapping which sends every open (resp. closed) set of X in an open (resp. closed) set of Y and viceversa, i.e. an homeomorphism is both an open and closed map. Note that the homeomorphism gives an equivalence relation on the class of all topological spaces.

Examples 1.1.36. *In these examples we consider any subset of* $\mathbb{R}^n$ *endowed with the subset topology induced by the Euclidean topology on* $\mathbb{R}^n$*.*

1. *Any open interval of* $\mathbb{R}$ *is homeomorphic to any other open interval of* $\mathbb{R}$ *and also to* $\mathbb{R}$ *itself.*
2. *A circle and a square in* $\mathbb{R}^2$ *are homeomorphic.*
3. *The circle* S^1 *with a point removed is homeomorphic to* $\mathbb{R}$*.*

Let us consider now the case when a set X carries two different topologies τ_1 and τ_2. Then the following two properties are equivalent:

- the identity ι of X is continuous as a mapping from (X, τ_1) and (X, τ_2)
- the topology τ_1 is finer than the topology τ_2.

Therefore, ι is a homeomorphism if and only if the two topologies coincide.

Proof. Suppose that ι is continuous. Let $U \in \tau_2$. Then $\iota^{-1}(U) = U \in \tau_1$, hence $U \in \tau_1$. Therefore, $\tau_2 \subseteq \tau_1$. Conversely, assume that $\tau_2 \subseteq \tau_1$ and take any $U \in \tau_2$. Then $U \in \tau_1$ and by definition of identity we know that $\iota^{-1}(U) = U$. Hence, $\iota^{-1}(U) \in \tau_1$ and therefore, ι is continuous. $\square$

Proposition 1.1.37. *Continuous maps preserve the convergence of sequences. That is, if $f : X \to Y$ is a continuous map between two topological spaces (X, τ_X) and (Y, τ_Y) and if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of points in X convergent to a point $x \in X$ then $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x) \in Y$.*

Proof. (Sheet 2, Exercise 4 b)) □

1.1.5 Hausdorff spaces

Definition 1.1.38. *A topological space X is said to be* Hausdorff *(or* separated*) if any two distinct points of X have neighbourhoods without common points; or equivalently if:*
(T2) two distinct points always lie in disjoint open sets.

In literature, the Hausdorff space are often called *T2-spaces* and the axiom (T2) is said to be the *separation axiom.*

Proposition 1.1.39. *In a Hausdorff space the intersection of all closed neighbourhoods of a point contains the point alone. Hence, the singletons are closed.*

Proof. Let us fix a point $x \in X$, where X is a Hausdorff space. Take $y \neq x$. By definition, there exist a neighbourhood $U(x)$ of x and a neighbourhood $V(y)$ of y s.t. $U(x) \cap V(y) = \emptyset$. Therefore, $y \notin \overline{U(x)}$. □

Examples 1.1.40.

1. *Any metric space is Hausdorff.*
   Indeed, for any $x, y \in (X, d)$ with $x \neq y$ just choose $0 < \varepsilon < \frac{1}{2}d(x, y)$ and you get $B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset$.
2. *Any set endowed with the discrete topology is a Hausdorff space.*
   Indeed, any singleton is open in the discrete topology so for any two distinct point x, y we have that $\{x\}$ and $\{y\}$ are disjoint and open.
3. *The only Hausdorff topology on a finite set is the discrete topology.*
   In fact, since X is finite, any subset S of X is finite and so S is a finite union of singletons. But since X is also Hausdorff, the previous proposition implies that any singleton is closed. Hence, any subset S of X is closed and so the topology on X has to be the discrete one.
4. *An infinite set with the cofinite topology is not Hausdorff.*
   In fact, any two non-empty open subsets O_1, O_2 in the cofinite topology on X are complements of finite subsets. Therefore, their intersection $O_1 \cap O_2$ is still a complement of a finite subset, but X is infinite and so $O_1 \cap O_2 \neq \emptyset$. Hence, X is not Hausdorff.

1.2 Linear mappings between vector spaces

The basic notions from linear algebra are assumed to be well-known and so they are not recalled here. However, we briefly give again the definition of vector space and fix some general terminology for linear mappings between vector spaces. In this section we are going to consider vector spaces over the field $\mathbb{K}$ of real or complex numbers which is given the usual euclidean topology defined by means of the modulus.

Definition 1.2.1. *A set X with the two mappings:*

$$\begin{array}{rcll} X \times X & \to & X & \\ (x,y) & \mapsto & x+y & \textit{vector addition} \end{array}$$

$$\begin{array}{rcll} \mathbb{K} \times X & \to & X & \\ (\lambda,x) & \mapsto & \lambda x & \textit{scalar multiplication} \end{array}$$

is a vector space *(or* linear space*) over $\mathbb{K}$ if the following axioms are satisfied:*

(L1) *1.* $(x+y)+z = x+(y+z)$, $\forall\, x,y,z \in X$ *(associativity of +)*

2. $x+y = y+x$, $\forall\, x,y \in X$ *(commutativity of +)*

3. $\exists\, o \in X$*:* $x+o = x$, $\forall\, x, \in X$ *(neutral element for +)*

4. $\forall\, x \in X$, $\exists! -x \in X$ *s.t.* $x+(-x) = o$ *(inverse element for +)*

(L2) *1.* $\lambda(\mu x) = (\lambda\mu)x$, $\forall\, x \in X$, $\forall\, \lambda,\mu \in \mathbb{K}$
(compatibility of scalar multiplication with field multiplication)

2. $1x = x\, \forall\, x \in X$ *(neutral element for scalar multiplication)*

3. $(\lambda+\mu)x = \lambda x + \mu x$, $\forall\, x \in X$, $\forall\, \lambda,\mu \in \mathbb{K}$
(distributivity of scalar multiplication with respect to field addition)

4. $\lambda(x+y) = \lambda x + \lambda y$, $\forall\, x,y \in X$, $\forall\, \lambda \in \mathbb{K}$
(distributivity of scalar multiplication wrt vector addition)

Definition 1.2.2.
Let X, Y be two vector space over $\mathbb{K}$. A mapping $f : X \to Y$ is called linear mapping *or* homomorphism *if f preserves the vector space structure, i.e.* $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)\, \forall\, x,y \in X$, $\forall\, \lambda,\mu \in \mathbb{K}$.

Definition 1.2.3.

- *A linear mapping from X to itself is called* endomorphism*.*
- *A one-to-one linear mapping is called* monomorphism*. If S is a subspace of X, the identity map is a monomorphism and it is called* embedding*.*
- *An onto (surjective) linear mapping is called* epimorphism*.*
- *A bijective (one-to-one and onto) linear mapping between two vector spaces X and Y over $\mathbb{K}$ is called* (algebraic) isomorphism*. If such a map exists, we say that X and Y are (algebraically) isomorphic $X \cong Y$.*
- *An isomorphism from X into itself is called* automorphism*.*

It is easy to prove that: A linear mapping is one-to-one (injective) if and only if $f(x) = 0$ implies $x = 0$.

Definition 1.2.4. *A linear mapping from* $X \to \mathbb{K}$ *is called* linear functional *or* linear form *on* X*. The set of all linear functionals on* X *is called* algebraic dual *and it is denoted by* X^**.*

Note that the dual space of a finite dimensional vector space X is isomorphic to X.

Chapter 2

Topological Vector Spaces

2.1 Definition and properties of a topological vector space

In this section we are going to consider vector spaces over the field $\mathbb{K}$ of real or complex numbers which is given the usual euclidean topology defined by means of the modulus.

Definition 2.1.1. *A vector space X over $\mathbb{K}$ is called a* topological vector space *(t.v.s.) if X is provided with a topology τ which is compatible with the vector space structure of X, i.e. τ makes the vector space operations both continuous.*

More precisely, the condition in the definition of t.v.s. requires that:

$$\begin{array}{ccc} X \times X & \to & X \\ (x, y) & \mapsto & x + y \end{array} \qquad \textit{vector addition}$$

$$\begin{array}{ccc} \mathbb{K} \times X & \to & X \\ (\lambda, x) & \mapsto & \lambda x \end{array} \qquad \textit{scalar multiplication}$$

are both continuous when we endow X with the topology τ, $\mathbb{K}$ with the euclidean topology, $X \times X$ and $\mathbb{K} \times X$ with the correspondent product topologies.

Remark 2.1.2. *If (X, τ) is a t.v.s then it is clear from Definition 2.1.1 that $\sum_{k=1}^{N} \lambda_k^{(n)} x_k^{(n)} \to \sum_{k=1}^{N} \lambda_k x_k$ as $n \to \infty$ w.r.t. τ if for each $k = 1, \ldots, N$ as $n \to \infty$ we have that $\lambda_k^{(n)} \to \lambda_k$ w.r.t. the euclidean topology on $\mathbb{K}$ and $x_k^{(n)} \to x_k$ w.r.t. τ.*

Let us discuss now some examples and counterexamples of t.v.s.

Examples 2.1.3.

a) Every vector space X over $\mathbb{K}$ endowed with the trivial topology is a t.v.s..

b) Every normed vector space endowed with the topology given by the metric induced by the norm is a t.v.s. (Sheet 2, Exercise 1 a)).

c) There are also examples of spaces whose topology cannot be induced by a norm or a metric but that are t.v.s., e.g. the space of infinitely differentiable functions, the spaces of test functions and the spaces of distributions (we will see later in details their topologies).

In general, a metric vector space is not a t.v.s.. Indeed, there exist metrics for which both the vector space operations of sum and product by scalars are discontinuous (see Sheet 3, Exercise 1 c) for an example).

Proposition 2.1.4. *Every vector space X over $\mathbb{K}$ endowed with the discrete topology is not a t.v.s. unless $X = \{o\}$.*

Proof. Assume by a contradiction that it is a t.v.s. and take $o \neq x \in X$. The sequence $\alpha_n = \frac{1}{n}$ in $\mathbb{K}$ converges to 0 in the euclidean topology. Therefore, since the scalar multiplication is continuous, $\alpha_n x \to o$, i.e. for any neighbourhood U of o in X there exists $m \in \mathbb{N}$ s.t. $\alpha_n x \in U$ for all $n \geq m$. In particular, we can take $U = \{o\}$ since it is itself open in the discrete topology. Hence, $\alpha_m x = o$, which implies that $x = o$ and so a contradiction. □

Definition 2.1.5. *Two t.v.s. X and Y over $\mathbb{K}$ are* (topologically) isomorphic *if there exists a vector space isomorphism $X \to Y$ which is at the same time a homeomorphism (i.e. bijective, linear, continuous and inverse continuous).*

In analogy to Definition 1.2.3, let us collect here the corresponding terminology for mappings between two t.v.s..

Definition 2.1.6. *Let X and Y be two t.v.s. on $\mathbb{K}$.*

- *A* topological homomorphism *from X to Y is a linear mapping which is also continuous and open.*
- *A* topological monomorphism *from X to Y is an injective topological homomorphism.*
- *A* topological isomorphism *from X to Y is a bijective topological homomorphism.*
- *A* topological automorphism *of X is a topological isomorphism from X into itself.*

Proposition 2.1.7. *Given a t.v.s. X, we have that:*

1. *For any $x_0 \in X$, the mapping $x \mapsto x + x_0$ (*translation *by x_0) is a homeomorphism of X onto itself.*
2. *For any $0 \neq \lambda \in \mathbb{K}$, the mapping $x \mapsto \lambda x$ (*dilation *by λ) is a topological automorphism of X.*

Proof. Both mappings are continuous by the very definition of t.v.s.. Moreover, they are bijections by the vector space axioms and their inverses $x \mapsto x - x_0$ and $x \mapsto \frac{1}{\lambda}x$ are also continuous. Note that the second map is also linear so it is a topological automorphism. $\square$

Proposition 2.1.7–1 shows that the topology of a t.v.s. is always a *translation invariant topology*, i.e. all translations are homeomorphisms. Note that the translation invariance of a topology τ on a vector space X is not sufficient to conclude (X, τ) is a t.v.s..

Example 2.1.8. *If a metric d on a vector space X is translation invariant, i.e. $d(x+z, y+z) = d(x, y)$ for all $x, y \in X$ (e.g. the metric induced by a norm), then the topology induced by the metric is translation invariant and the addition is always continuous. However, the multiplication by scalars does not need to be necessarily continuous (take d to be the discrete metric, then the topology generated by the metric is the discrete topology which is not compatible with the scalar multiplication see Proposition 2.1.4).*

The translation invariance of the topology of a t.v.s. means, roughly speaking, that a t.v.s. X topologically looks about any point as it does about any other point. More precisely:

Corollary 2.1.9. *The filter $\mathcal{F}(x)$ of neighbourhoods of $x \in X$ coincides with the family of the sets $O + x$ for all $O \in \mathcal{F}(o)$, where $\mathcal{F}(o)$ is the filter of neighbourhoods of the origin o (i.e. the neutral element of the vector addition).*

Proof. (Sheet 3, Exercise 2 a)) $\square$

Thus the topology of a t.v.s. is completely determined by the filter of neighbourhoods of any of its points, in particular by the filter of neighbourhoods of the origin o or, more frequently, by a base of neighbourhoods of the origin o. Therefore, we need some criteria on a filter of a vector space X which ensures that it is the filter of neighbourhoods of the origin w.r.t. some topology compatible with the vector structure of X.

Theorem 2.1.10. *A filter $\mathcal{F}$ of a vector space X over $\mathbb{K}$ is the filter of neighbourhoods of the origin w.r.t. some topology compatible with the vector structure of X if and only if*

1. *The origin belongs to every set $U \in \mathcal{F}$*
2. *$\forall U \in \mathcal{F}$, $\exists V \in \mathcal{F}$ s.t. $V + V \subset U$*
3. *$\forall U \in \mathcal{F}$, $\forall \lambda \in \mathbb{K}$ with $\lambda \neq 0$ we have $\lambda U \in \mathcal{F}$*
4. *$\forall U \in \mathcal{F}$, U is absorbing.*
5. *$\forall U \in \mathcal{F}$, $\exists V \in \mathcal{F}$ s.t. $V \subset U$ is balanced.*

Before proving the theorem, let us fix some definitions and notations:

Definition 2.1.11. *Let U be a subset of a vector space X.*

1. *U is* absorbing *(or* radial*) if $\forall x \in X$ $\exists \rho > 0$ s.t. $\forall \lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$ we have $\lambda x \in U$. Roughly speaking, we may say that a subset is absorbing if it can be made by dilation to swallow every point of the whole space.*
2. *U is* balanced *(or* circled*) if $\forall x \in U$, $\forall \lambda \in \mathbb{K}$ with $|\lambda| \leq 1$ we have $\lambda x \in U$. Note that the line segment joining any point x of a balanced set U to $-x$ lies in U.*

Clearly, o must belong to every absorbing or balanced set. The underlying field can make a substantial difference. For example, if we consider the closed interval $[-1, 1] \subset \mathbb{R}$ then this is a balanced subset of $\mathbb{C}$ as real vector space, but if we take $\mathbb{C}$ as complex vector space then it is not balanced. Indeed, if we take $i \in \mathbb{C}$ we get that $i1 = i \notin [-1, 1]$.

Examples 2.1.12.

a) *In a normed space the unit balls centered at the origin are absorbing and balanced.*

b) *The unit ball B centered at $(\frac{1}{2}, 0) \in \mathbb{R}^2$ is absorbing but not balanced in the real vector space $\mathbb{R}^2$ endowed with the euclidean norm.* Indeed, B is a neighbourhood of the origin and so by Theorem 2.1.10-4 is absorbing. However, B is not balanced because for example if we take $x = (1, 0) \in B$ and $\lambda = -1$ then $\lambda x \notin B$.

c) *In the real vector space $\mathbb{R}^2$ endowed with the euclidean topology, the subset in Figure 2.1 is absorbing and the one in Figure 2.2 is balanced.*

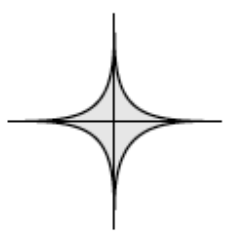

Figure 2.1: Absorbing

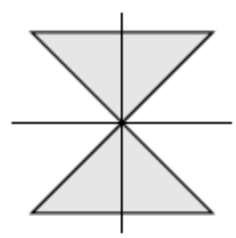

Figure 2.2: Balanced

d) *The polynomials $\mathbb{R}[x]$ are a balanced but not absorbing subset of the real space $\mathcal{C}([0, 1], \mathbb{R})$ of continuous real valued functions on* $[0, 1]$. Indeed, any multiple of a polynomial is still a polynomial but not every continuous function can be written as multiple of a polynomial.

e) *The subset $A := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq |z_2|\}$ of the complex space $\mathbb{C}^2$ with the euclidean topology is balanced but $\mathring{A}$ is not balanced.*

Proposition 2.1.13.
a) If B is a balanced subset of a t.v.s. X then so is $\bar{B}$.
b) If B is a balanced subset of a t.v.s. X and $o \in \mathring{B}$ then $\mathring{B}$ is balanced.

Proof. (Sheet 3, Exercise 2 b) c)) □

Proof. of Theorem 2.1.10.
Necessity part.
Suppose that X is a t.v.s. then we aim to show that the filter of neighbourhoods of the origin $\mathcal{F}$ satisfies the properties 1,2,3,4,5. Let $U \in \mathcal{F}$.

1. obvious, since every set $U \in \mathcal{F}$ is a neighbourhood of the origin o.
2. Since by the definition of t.v.s. the addition $(x, y) \mapsto x+y$ is a continuous mapping, the preimage of U under this map must be a neighbourhood of $(o, o) \in X \times X$. Therefore, it must contain a rectangular neighbourhood $W \times W'$ where $W, W' \in \mathcal{F}$. Taking $V = W \cap W'$ we get the conclusion, i.e. $V + V \subset U$.
3. By Proposition 2.1.7, fixed an arbitrary $0 \neq \lambda \in \mathbb{K}$, the map $x \mapsto \lambda^{-1}x$ of X into itself is continuous. Therefore, the preimage of any neighbourhood U of the origin must be also such a neighbourhood. This preimage is clearly λU, hence $\lambda U \in \mathcal{F}$.
4. Suppose by contradiction that U is not absorbing. Then there exists $y \in X$ such that $\forall n \in \mathbb{N}$ we have that $\frac{1}{n}y \notin U$. This contradicts the convergence of $\frac{1}{n}y \to o$ as $n \to \infty$ (because $U \in \mathcal{F}$ must contain infinitely many terms of the sequence $(\frac{1}{n}y)_{n\in\mathbb{N}}$.
5. Since by the definition of t.v.s. the scalar multiplication $\mathbb{K} \times X \to X$, $(\lambda, x) \mapsto \lambda x$ is continuous, the preimage of U under this map must be a neighbourhood of $(0, o) \in \mathbb{K} \times X$. Therefore, it contains a rectangular neighbourhood $N \times W$ where N is a neighbourhood of 0 in the euclidean topology on $\mathbb{K}$ and $W \in \mathcal{F}$. On the other hand, there exists $\rho > 0$ such that $B_\rho(0) := \{\lambda \in \mathbb{K} : |\lambda| \leq \rho\} \subseteq N$. Thus $B_\rho(0) \times W$ is contained in the preimage of U under the scalar multiplication, i.e. $\lambda W \subset U$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$. Hence, the set $V = \cup_{|\lambda|\leq\rho}\lambda W \subset U$. Now $V \in \mathcal{F}$ since each $\lambda W \in \mathcal{F}$ by 3 and V is clearly balanced (since for any $x \in V$ there exists $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$ s.t. $x \in \lambda W$ and therefore for any $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$ we get $\alpha x \in \alpha\lambda W \subset V$ because $|\alpha\lambda| \leq \rho$).

Sufficiency part.
Suppose that the conditions 1,2,3,4,5 hold for a filter $\mathcal{F}$ of the vector space X. We want to show that there exists a topology τ on X such that $\mathcal{F}$ is the filter of neighbourhoods of the origin w.r.t. to τ and (X, τ) is a t.v.s. according to Definition 2.1.1.

Let us define for any $x \in X$ the filter $\mathcal{F}(x) := \{U + x : U \in \mathcal{F}\}$. It is easy to see that $\mathcal{F}(x)$ fulfills the properties (N1) and (N2) of Theorem 1.1.9. In fact, we have:

- By 1 we have that $\forall U \in \mathcal{F}$, $o \in U$, then $\forall U \in \mathcal{F}$, $x = o + x \in U + x$, i.e. $\forall A \in \mathcal{F}(x)$, $x \in A$.
- Let $A \in \mathcal{F}(x)$ then $A = U + x$ for some $U \in \mathcal{F}$. By 2, we have that there exists $V \in \mathcal{F}$ s.t. $V + V \subset U$. Define $B := V + x \in \mathcal{F}(x)$ and take any $y \in B$ then we have $V + y \subset V + B \subset V + V + x \subset U + x = A$. But $V + y$ belongs to the filter $\mathcal{F}(y)$ and therefore so does A.

By Theorem 1.1.9, there exists a unique topology τ on X such that $\mathcal{F}(x)$ is the filter of neighbourhoods of each point $x \in X$ and so for which in particular $\mathcal{F}$ is the filter of neighbourhoods of the origin.

It remains to prove that the vector addition and the scalar multiplication in X are continuous w.r.t. to τ.

- The continuity of the addition easily follows from the property 2. Indeed, let $(x_0, y_0) \in X \times X$ and take a neighbourhood W of its image $x_0 + y_0$. Then $W = U + x_0 + y_0$ for some $U \in \mathcal{F}$. By 2, there exists $V \in \mathcal{F}$ s.t. $V + V \subset U$ and so $(V + x_0) + (V + y_0) \subset W$. This implies that the preimage of W under the addition contains $(V + x_0) \times (V + y_0)$ which is a neighbourhood of (x_0, y_0).
- To prove the continuity of the scalar multiplication, let $(\lambda_0, x_0) \in \mathbb{K} \times X$ and take a neighbourhood U' of $\lambda_0 x_0$. Then $U' = U + \lambda_0 x_0$ for some $U \in \mathcal{F}$. By 2 and 5, there exists $W \in \mathcal{F}$ s.t. $W + W + W \subset U$ and W is balanced. By 4, W is also absorbing so there exists $\rho > 0$ (w.l.o.g we can take $\rho \leq 1$) such that $\forall \lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$ we have $\lambda x_0 \in W$.

  Suppose $\lambda_0 = 0$ then $\lambda_0 x_0 = o$ and $U' = U$. Now
  $$Im(B_\rho(0) \times (W + x_0)) = \{\lambda y + \lambda x_0 : \lambda \in B_\rho(0), y \in W\}.$$
  As $\lambda \in B_\rho(0)$ and W is absorbing, $\lambda x_0 \in W$. Also since $|\lambda| \leq \rho \leq 1$ for all $\lambda \in B_\rho(0)$ and since W is balanced, we have $\lambda W \subset W$. Thus $Im(B_\rho(0) \times (W + x_0)) \subset W + W \subset W + W + W \subset U$ and so the preimage of U under the scalar multiplication contains $B_\rho(0) \times (W + x_0)$ which is a neighbourhood of $(0, x_0)$.

  Suppose $\lambda_0 \neq 0$ and take $\sigma = \min\{\rho, |\lambda_0|\}$. Then $Im((B_\sigma(0) + \lambda_0) \times (|\lambda_0|^{-1} W + x_0)) = \{\lambda|\lambda_0|^{-1} y + \lambda x_0 + \lambda_0 |\lambda_0|^{-1} y + \lambda x_0 : \lambda \in B_\sigma(0), y \in W\}$. As $\lambda \in B_\sigma(0)$, $\sigma \leq \rho$ and W is absorbing, $\lambda x_0 \in W$. Also since $\forall \lambda \in B_\sigma(0)$ the modulus of $\lambda|\lambda_0|^{-1}$ and $\lambda_0|\lambda_0|^{-1}$ are both ≤ 1 and since W is balanced, we have $\lambda|\lambda_0|^{-1} W, \lambda_0|\lambda_0|^{-1} W \subset W$. Thus $Im(B_\sigma(0) + \lambda_0 \times (|\lambda_0|^{-1} W + x_0)) \subset W + W + W + \lambda_0 x_0 \subset U + \lambda_0 x_0$ and so the preimage of $U + \lambda_0 x_0$ under the scalar multiplication contains $B_\sigma(0) + \lambda_0 \times (|\lambda_0|^{-1} W + x_0)$ which is a neighbourhood of (λ_0, x_0). $\square$

It easily follows from previous theorem that:

Corollary 2.1.14.

a) Every t.v.s. has always a base of closed neighbourhoods of the origin.

b) Every t.v.s. has always a base of balanced absorbing neighbourhoods of the origin. In particular, it has always a base of closed balanced absorbing neighbourhoods of the origin.

c) Proper subspaces of a t.v.s. are never absorbing. In particular, if M is an open subspace of a t.v.s. X then $M = X$.

Proof. (Sheet 3, Exercise 3) □

Let us show some further useful properties of the t.v.s.:

Proposition 2.1.15.

1. *Every linear subspace of a t.v.s. endowed with the correspondent subspace topology is itself a t.v.s..*
2. *The closure $\overline{H}$ of a linear subspace H of a t.v.s. X is again a linear subspace of X.*
3. *Let X, Y be two t.v.s. and $f : X \to Y$ a linear map. f is continuous if and only if f is continuous at the origin o.*

Proof.

1. This clearly follows by the fact that the addition and the multiplication restricted to the subspace are just a composition of continuous maps (recall that inclusion is continuous in the subspace topology c.f. Definition 1.1.17).
2. Let $x_0, y_0 \in \overline{H}$ and let us take any $U \in \mathcal{F}(o)$. By Theorem 2.1.10-2, there exists $V \in \mathcal{F}(o)$ s.t. $V + V \subset U$. Then, by definition of closure points, there exist $x, y \in H$ s.t. $x \in V + x_0$ and $y \in V + y_0$. Therefore, we have that $x + y \in H$ (since H is a linear subspace) and $x+y \in (V+x_0)+(V+y_0) \subset U+x_0+y_0$. Hence, $x_0+y_0 \in \overline{H}$. Similarly, one can prove that if $x_0 \in \overline{H}$, $\lambda x_0 \in \bar{H}$ for any $\lambda \in \mathbb{K}$.
3. Assume that f is continuous at $o \in X$ and fix any $x \neq o$ in X. Let U be an arbitrary neighbourhood of $f(x) \in Y$. By Corollary 2.1.9, we know that $U = f(x) + V$ where V is a neighbourhood of $o \in Y$. Since f is linear we have that:

$$f^{-1}(U) = f^{-1}(f(x) + V) \supset x + f^{-1}(V).$$

By the continuity at the origin of X, we know that $f^{-1}(V)$ is a neighbourhood of $o \in X$ and so $x + f^{-1}(V)$ is a neighbourhood of $x \in X$.

□

2.2 Hausdorff topological vector spaces

For convenience let us recall here the definition of Hausdorff space.

Definition 2.2.1. *A topological space X is said to be* Hausdorff *or* (T2) *if any two distinct points of X have neighbourhoods without common points; or equivalently if two distinct points always lie in disjoint open sets.*

Note that in a Hausdorff space, any set consisting of a single point is closed but there are topological spaces with the same property which are not Hausdorff and we will see in this section that such spaces are not t.v.s..

Definition 2.2.2. *A topological space X is said to be* (T1) *if, given two distinct points of X, each lies in a neighborhood which does not contain the other point; or equivalently if, for any two distinct points, each of them lies in an open subset which does not contain the other point.*

It is easy to see that in a topological space which is (T1) all singletons are closed (Sheet 4, Exercise 2).

From the definition it is clear that (T2) implies (T1) but in general the inverse does not hold (c.f. Examples 1.1.40-4 for an example of topological space which is T1 but not T2). However, the following results shows that a t.v.s is Hausdorff if and only if it is (T1).

Proposition 2.2.3. *A t.v.s. X is Hausdorff iff*

$$\forall\, o \neq x \in X,\, \exists\, U \in \mathcal{F}(o) \; s.t. \;\, x \notin U. \tag{2.1}$$

Proof.

($\Rightarrow$) Let (X, τ) be Hausdorff. Then there exist $U \in \mathcal{F}(o)$ and $V \in \mathcal{F}(x)$ s.t. $U \cap V = \emptyset$. This means in particular that $x \notin U$.

($\Leftarrow$) Assume that (2.1) holds and let $x, y \in X$ with $x \neq y$, i.e. $x - y \neq o$. Then there exists $U \in \mathcal{F}(o)$ s.t. $x - y \notin U$. By (2) and (5) of Theorem 2.1.10, there exists $V \in \mathcal{F}(o)$ balanced and s.t. $V + V \subset U$. Since V is balanced $V = -V$ then we have $V - V \subset U$. Suppose now that $(V + x) \cap (V + y) \neq \emptyset$, then there exists $z \in (V + x) \cap (V + y)$, i.e. $z = v + x = w + y$ for some $v, w \in V$. Then $x - y = w - v \in V - V \subset U$ and so $x - y \in U$ which is a contradiction. Hence, $(V + x) \cap (V + y) = \emptyset$ and by Corollary 2.1.9 we know that $V + x \in \mathcal{F}(x)$ and $V + y \in \mathcal{F}(y)$. Hence, X is (T2). $\square$

Note that since the topology of a t.v.s. is translation invariant then the previous proposition guarantees that *a t.v.s is Hausdorff iff it is (T1).* As a matter of fact, we have the following result:

Corollary 2.2.4. *For a t.v.s. X the following are equivalent:*
a) X is Hausdorff.
b) the intersection of all neighbourhoods of the origin o is just $\{o\}$.
c) $\{0\}$ is closed.

Note that in a t.v.s. $\{0\}$ is closed is equivalent to say that all singletons are closed (and so that the space is (T1)).

Proof.
a)$\Rightarrow$ b) Let X be a Hausdorff t.v.s. space. Clearly, $\{o\} \subseteq \cap_{U\in\mathcal{F}(o)}U$. Now if b) does not hold, then there exists $x \in \cap_{U\in\mathcal{F}(o)}U$ with $x \neq o$. But by the previous theorem we know that (2.1) holds and so there exists $V \in \mathcal{F}(o)$ s.t. $x \notin V$ and so $x \notin \cap_{U\in\mathcal{F}(o)}U$ which is a contradiction.

b)$\Rightarrow$ c) Assume that $\cap_{U\in\mathcal{F}(o)}U = \{o\}$. If $x \in \overline{\{o\}}$ then $\forall\, V_x \in \mathcal{F}(x)$ we have $V_x \cap \{o\} \neq \emptyset$, i.e. $o \in V_x$. By Corollary 2.1.9 we know that each $V_x = U + x$ with $U \in \mathcal{F}(o)$. Then $o = u + x$ for some $u \in U$ and so $x = -u \in -U$. This means that $x \in \cap_{U\in\mathcal{F}(o)}(-U)$. Since every dilation is an homeomorphism and b) holds, we have that $x \in \cap_{U\in\mathcal{F}(o)}U = \{0\}$. Hence, $x = 0$ and so $\overline{\{o\}} = \{o\}$, i.e. $\{o\}$ is closed.

c)$\Rightarrow$ a) Assume that X is not Hausdorff. Then by the previous proposition (2.1) does not hold, i.e. there exists $x \neq o$ s.t. $x \in U$ for all $U \in \mathcal{F}(o)$. This means that $x \in \cap_{U\in\mathcal{F}(o)}U \subseteq \cap_{U\in\mathcal{F}(o)\text{closed}}U = \overline{\{o\}}$ By c), $\overline{\{o\}} = \{o\}$ and so $x = 0$ which is a contradiction. $\square$

Example 2.2.5. *Every vector space with an infinite number of elements endowed with the cofinite topology is not a tvs.* It is clear that in such topological space all singletons are closed (i.e. it is T1). Therefore, if it was a t.v.s. then by the previous results it should be a Hausdorff space which is not true as showed in Example 1.1.40.

2.3 Quotient topological vector spaces

Quotient topology

Let X be a topological space and $\sim$ be any equivalence relation on X. Then the *quotient set* $X/\sim$ is defined to be the set of all equivalence classes w.r.t. to $\sim$. The map $\phi : X \to X/\sim$ which assigns to each $x \in X$ its equivalence class $\phi(x)$ w.r.t. $\sim$ is called the *canonical map* or *quotient map*. Note that ϕ is surjective. We may define a topology on $X/\sim$ by setting that: a subset U of $X/\sim$ is open iff the preimage $\phi^{-1}(U)$ is open in X. This is called the *quotient topology* on $X/\sim$. Then it is easy to verify (Sheet 4, Exercise 2) that:

- the quotient map ϕ is continuous.
- the quotient topology on $X/\sim$ is the finest topology on $X/\sim$ s.t. ϕ is continuous.

Note that the quotient map ϕ is not necessarily open or closed.

Example 2.3.1. *Consider $\mathbb{R}$ with the standard topology given by the modulus and define the following equivalence relation on $\mathbb{R}$:*

$$x \sim y \Leftrightarrow (x = y \vee \{x, y\} \subset \mathbb{Z}).$$

Let $\mathbb{R}/\sim$ be the quotient set w.r.t $\sim$ and $\phi : \mathbb{R} \to \mathbb{R}/\sim$ the correspondent quotient map. Let us consider the quotient topology on $\mathbb{R}/\sim$. Then ϕ is not an open map. In fact, if U is an open proper subset of $\mathbb{R}$ containing an integer, then $\phi^{-1}(\phi(U)) = U \cup \mathbb{Z}$ which is not open in $\mathbb{R}$ with the standard topology. Hence, $\phi(U)$ is not open in $\mathbb{R}/\sim$ with the quotient topology.

For an example of quotient map which is not closed see Example 2.3.3 in the following.

Quotient vector space

Let X be a vector space and M a linear subspace of X. For two arbitrary elements $x, y \in X$, we define $x \sim_M y$ iff $x - y \in M$. It is easy to see that $\sim_M$ is an equivalence relation: it is reflexive, since $x - x = 0 \in M$ (every linear subspace contains the origin); it is symmetric, since $x - y \in M$ implies $-(x - y) = y - x \in M$ (if a linear subspace contains an element, it contains its inverse); it is transitive, since $x - y \in M$, $y - z \in M$ implies $x - z = (x - y) + (y - z) \in M$ (when a linear subspace contains two vectors, it also contains their sum). Then X/M is defined to be the quotient set $X/\sim_M$, i.e. the set of all equivalence classes for the relation $\sim_M$ described above. The canonical (or quotient) map $\phi : X \to X/M$ which assigns to each $x \in X$ its equivalence class $\phi(x)$ w.r.t. the relation $\sim_M$ is clearly surjective. Using the fact that M is a linear subspace of X, it is easy to check that:

1. if $x \sim_M y$, then $\forall\, \lambda \in \mathbb{K}$ we have $\lambda x \sim_M \lambda y$.
2. if $x \sim_M y$, then $\forall\, z \in X$ we have $x + z \sim_M y + z$.

These two properties guarantee that the following operations are well-defined on X/M:

- vector addition: $\forall\, \phi(x), \phi(y) \in X/M$, $\phi(x) + \phi(y) := \phi(x + y)$
- scalar multiplication: $\forall\, \lambda \in \mathbb{K}$, $\forall\, \phi(x) \in X/M$, $\lambda\phi(x) := \phi(\lambda x)$

X/M with the two operations defined above is a vector space and therefore it is often called *quotient vector space*. Then the quotient map ϕ is clearly linear.

Quotient topological vector space

Let X be now a t.v.s. and M a linear subspace of X. Consider the quotient vector space X/M and the quotient map $\phi : X \to X/M$ defined in Section 2.3. Since X is a t.v.s, it is in particular a topological space, so we can consider on X/M the quotient topology defined in Section 2.3. We already know that in this topological setting ϕ is continuous but actually the structure of t.v.s. on X guarantees also that it is open.

Proposition 2.3.2. *For a linear subspace M of a t.v.s. X, the quotient mapping $\phi : X \to X/M$ is open (i.e. carries open sets in X to open sets in X/M) when X/M is endowed with the quotient topology.*

Proof. Let V open in X. Then we have

$$\phi^{-1}(\phi(V)) = V + M = \cup_{m\in M}(V + m)$$

Since X is a t.v.s, its topology is translation invariant and so $V + m$ is open for any $m \in M$. Hence, $\phi^{-1}(\phi(V))$ is open in X as union of open sets. By definition, this means that $\phi(V)$ is open in X/M endowed with the quotient topology. □

It is then clear that ϕ carries neighborhoods of a point in X into neighborhoods of a point in X/M and viceversa. Hence, the neighborhoods of the origin in X/M are direct images under ϕ of the neighborhoods of the origin in X. In conclusion, when X is a t.v.s and M is a subspace of X, we can rewrite the definition of quotient topology on X/M in terms of neighborhoods as follows: *the filter of neighborhoods of the origin of X/M is exactly the image under ϕ of the filter of neighborhoods of the origin in X.*

It is not true, in general (not even when X is a t.v.s. and M is a subspace of X), that the quotient map is closed.

Example 2.3.3.
Consider $\mathbb{R}^2$ with the euclidean topology and the hyperbola $H := \{(x, y) \in \mathbb{R}^2 : xy = 1\}$. If M is one of the coordinate axes, then $\mathbb{R}^2/M$ can be identified with the other coordinate axis and the quotient map ϕ with the orthogonal projection on it. All these identifications are also valid for the topologies. The hyperbola H is closed in $\mathbb{R}^2$ but its image under ϕ is the complement of the origin on a straight line which is open.

Corollary 2.3.4. *For a linear subspace M of a t.v.s. X, the quotient space X/M endowed with the quotient topology is a t.v.s..*

Proof.
For convenience, we denote here by A the vector addition in X/M and just by $+$ the vector addition in X. Let W be a neighbourhood of the origin o in X/M. We aim to prove that $A^{-1}(W)$ is a neighbourhood of (o, o) in $X/M \times X/M$.

The continuity of the quotient map $\phi : X \to X/M$ implies that $\phi^{-1}(W)$ is a neighbourhood of the origin in X. Then, by Theorem 2.1.10-2 (we can apply the theorem because X is a t.v.s.), there exists V neighbourhood of the origin in X s.t. $V + V \subseteq \phi^{-1}(W)$. Hence, by the linearity of ϕ, we get $A(\phi(V) \times \phi(V)) = \phi(V + V) \subseteq W$, i.e. $\phi(V) \times \phi(V) \subseteq A^{-1}(W)$. Since ϕ is also open, $\phi(V)$ is a neighbourhood of the origin o in X/M and so $A^{-1}(W)$ is a neighbourhood of (o, o) in $X/M \times X/M$.

A similar argument gives the continuity of the scalar multiplication. □

Proposition 2.3.5. *Let X be a t.v.s. and M a linear subspace of X. Consider X/M endowed with the quotient topology. Then the two following properties are equivalent:*
a) M is closed
b) X/M is Hausdorff

Proof.
In view of Corollary 2.2.4, (b) is equivalent to say that the complement of the origin in X/M is open w.r.t. the quotient topology. But the complement of the origin in X/M is exactly the image under ϕ of the complement of M in X. Since ϕ is an open continuous map, the image under ϕ of the complement of M in X is open in X/M iff the complement of M in X is open, i.e. (a) holds. □

Corollary 2.3.6. *If X is a t.v.s., then $X/\overline{\{o\}}$ endowed with the quotient topology is a Hausdorff t.v.s.. $X/\overline{\{o\}}$ is said to be the* Hausdorff t.v.s. associated *with the t.v.s. X. When a t.v.s. X is Hausdorff, X and $X/\overline{\{o\}}$ are topologically isomorphic.*

Proof.
Since X is a t.v.s. and $\{o\}$ is a linear subspace of X, $\overline{\{o\}}$ is a closed linear subspace of X. Then, by Corollary 2.3.4 and Proposition 2.3.5, $X/\overline{\{o\}}$ is a Hausdorff t.v.s.. If in addition X is Hausdorff, then Corollary 2.2.4 guarantees that $\overline{\{o\}} = \{o\}$ in X. Therefore, the quotient map $\phi : X \to X/\overline{\{o\}}$ is also injective because in this case $Ker(\phi) = \{o\}$. Hence, ϕ is a topological isomorphism (i.e. bijective, continuous, open, linear) between X and $X/\overline{\{o\}}$ which is indeed $X/\{o\}$. □

2.4 Continuous linear mappings between t.v.s.

Let X and Y be two vector spaces over $\mathbb{K}$ and $f : X \to Y$ a linear map. We define the *image* of f, and denote it by $Im(f)$, as the subset of Y:

$$Im(f) := \{y \in Y \,:\, \exists\, x \in X \text{ s.t. } y = f(x)\}.$$

We define the *kernel* of f, and denote it by $Ker(f)$, as the subset of X:

$$Ker(f) := \{x \in X \,:\, f(x) = 0\}.$$

Both $Im(f)$ and $Ker(f)$ are linear subspaces of Y and X, respectively. We have then the diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"] & Im(f) \arrow[r, "i"] & Y \\ X/Ker(f) \arrow[ur, "\bar{f}"] & & \end{tikzcd}$$

where i is the natural injection of $Im(f)$ into Y, i.e. the mapping which to each element y of $Im(f)$ assigns that same element y regarded as an element of Y; ϕ is the canonical map of X onto its quotient $X/Ker(f)$. The mapping $\bar{f}$ is defined so as to make the diagram commutative, which means that:

$$\forall\, x \in X,\; f(x) = \bar{f}(\phi(x)).$$

Note that

- $\bar{f}$ is well-defined.
  Indeed, if $\phi(x) = \phi(y)$, i.e. $x - y \in Ker(f)$, then $f(x-y) = 0$ that is $f(x) = f(y)$ and so $\bar{f}(\phi(x)) = \bar{f}(\phi(y))$.
- $\bar{f}$ is linear.
  This is an immediate consequence of the linearity of f and of the linear structure of $X/Ker(f)$.
- $\bar{f}$ is a one-to-one linear map of $X/Ker(f)$ onto $Im(f)$.
  The onto property is evident from the definition of $Im(f)$ and of $\bar{f}$. As for the one-to-one property, note that $\bar{f}(\phi(x)) = \bar{f}(\phi(y))$ means by definition that $f(x) = f(y)$, i.e. $f(x-y) = 0$. This is equivalent, by linearity of f, to say that $x - y \in Ker(f)$, which means that $\phi(x) = \phi(y)$.

The set of all linear maps (continuous or not) of a vector space X into another vector space Y is denoted by $\mathcal{L}(X;Y)$. Note that $\mathcal{L}(X;Y)$ is a vector space for the natural addition and multiplication by scalars of functions. Recall that when $Y = \mathbb{K}$, the space $\mathcal{L}(X;Y)$ is denoted by X^* and it is called the *algebraic dual* of X (see Definition 1.2.4).

Let us not turn to consider linear mapping between two t.v.s. X and Y. Since they posses a topological structure, it is natural to study in this setting continuous linear mappings.

Proposition 2.4.1. *Let $f : X \to Y$ a linear map between two t.v.s. X and Y. If Y is Hausdorff and f is continuous, then $Ker(f)$ is closed in X.*

Proof.
Clearly, $Ker(f) = f^{-1}(\{o\})$. Since Y is a Hausdorff t.v.s., $\{o\}$ is closed in Y and so, by the continuity of f, $Ker(f)$ is also closed in Y. □

Note that $Ker(f)$ might be closed in X also when Y is not Hausdorff. For instance, when $f \equiv 0$ or when f is injective and X is Hausdorff.

Proposition 2.4.2. *Let $f : X \to Y$ a linear map between two t.v.s. X and Y. The map f is continuous if and only if the map $\bar{f}$ is continuous.*

Proof.
Suppose f continuous and let U be an open subset in $Im(f)$. Then $f^{-1}(U)$ is open in X. By definition of $\bar{f}$, we have $\bar{f}^{-1}(U) = \phi(f^{-1}(U))$. Since the quotient map $\phi : X \to X/Ker(f)$ is open, $\phi(f^{-1}(U))$ is open in $X/Ker(f)$. Hence, $\bar{f}^{-1}(U)$ is open in $X/Ker(f)$ and so the map $\bar{f}$ is continuous. Viceversa, suppose that $\bar{f}$ is continuous. Since $f = \bar{f} \circ \phi$ and ϕ is continuous, f is also continuous as composition of continuous maps. □

In general, the inverse of $\bar{f}$, which is well defined on $Im(f)$ since $\bar{f}$ is injective, is not continuous. In other words, $\bar{f}$ is not necessarily bi-continuous.

The set of all continuous linear maps of a t.v.s. X into another t.v.s. Y is denoted by $L(X;Y)$ and it is a vector subspace of $\mathcal{L}(X;Y)$. When $Y = \mathbb{K}$, the space $L(X;Y)$ is usually denoted by X' which is called the *topological dual* of X, in order to underline the difference with X^* the algebraic dual of X. X' is a vector subspace of X^* and is exactly the vector space of all continuous linear functionals, or continuous linear forms, on X. The vector spaces X' and $L(X;Y)$ will play an important role in the forthcoming and will be equipped with various topologies.

2.5 Completeness for t.v.s.

This section aims to treat completeness for most general types of topological vector spaces, beyond the traditional metric framework. As well as in the case of metric spaces, we need to introduce the definition of a Cauchy sequence in a t.v.s..

Definition 2.5.1. *A sequence $S := \{x_n\}_{n\in\mathbb{N}}$ of points in a t.v.s. X is said to be a* Cauchy sequence *if*

$$\forall\, U \in \mathcal{F}(o) \,in\, X,\, \exists\, N \in \mathbb{N} :\, x_m - x_n \in U,\, \forall m, n \geq N. \tag{2.2}$$

This definition agrees with the usual one if the topology of X is defined by a translation-invariant metric d. Indeed, in this case, a basis of neighbourhoods of the origin is given by all the open balls centered at the origin. Therefore, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in such (X, d) iff $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$: $x_m - x_n \in B_\varepsilon(o)$, $\forall m, n \geq N$, i.e. $d(x_m, x_n) = d(x_m - x_n, o) < \varepsilon$.

By using the subsequences $S_m := \{x_n \in S \,:\, n \geq m\}$ of S, we can easily rewrite (2.2) in the following way

$$\forall\, U \in \mathcal{F}(o) \,\text{in}\, X,\, \exists\, N \in \mathbb{N} :\, S_N - S_N \subset U.$$

As we have already observed in Chapter 1, the collection $\mathcal{B} := \{S_m \,:\, m \in \mathbb{N}\}$ is a basis of the filter $\mathcal{F}_S$ associated with the sequence S. This immediately suggests what the definition of a Cauchy filter should be:

Definition 2.5.2. *A filter $\mathcal{F}$ on a subset A of a t.v.s. X is said to be a* Cauchy filter *if*

$$\forall\, U \in \mathcal{F}(o) \,in\, X,\, \exists\, M \subset A :\, M \in \mathcal{F} \text{ and } M - M \subset U.$$

In order to better illustrate this definition, let us come back to our reference example of a t.v.s. X whose topology is defined by a translation-invariant metric d. For any subset M of (X, d), recall that the diameter of M is defined as $diam(M) := \sup_{x,y\in M} d(x, y)$. Now if $\mathcal{F}$ is a Cauchy filter on X then, by definition, for any $\varepsilon > 0$ there exists $M \in \mathcal{F}$ s.t. $M - M \subset B_\varepsilon(o)$ and this simply means that $diam(M) \leq \varepsilon$. Therefore, Definition 2.5.2 can be rephrased in this case as follows: *a filter $\mathcal{F}$ on a subset A of such a metric t.v.s. X is a Cauchy filter if it contains subsets of A of arbitrarily small diameter.*

Going back to the general case, the following statement clearly holds.

Proposition 2.5.3. *The filter associated with a Cauchy sequence in a t.v.s. X is a Cauchy filter.*

Proposition 2.5.4.
Let X be a t.v.s.. Then the following properties hold:
a) The filter of neighborhoods of a point $x \in X$ is a Cauchy filter on X.
b) A filter finer than a Cauchy filter is a Cauchy filter.
c) Every converging filter is a Cauchy filter.

Proof.
a) Let $\mathcal{F}(x)$ be the filter of neighborhoods of a point $x \in X$ and let $U \in \mathcal{F}(o)$. By Theorem 2.1.10, there exists $V \in \mathcal{F}(o)$ such that $V - V \subset U$ and so such that $(V + x) - (V + x) \subset U$. Since X is a t.v.s., we know that $\mathcal{F}(x) = \mathcal{F}(o) + x$ and so $M := V + x \in \mathcal{F}(x)$. Hence, we have proved that for any $U \in \mathcal{F}(o)$ there exists $M \in \mathcal{F}(x)$ s.t. $M - M \subset U$, i.e. $\mathcal{F}(x)$ is a Cauchy filter.
b) Let $\mathcal{F}$ and $\mathcal{F}'$ be two filters of subsets of X such that $\mathcal{F}$ is a Cauchy filter and $\mathcal{F} \subseteq \mathcal{F}'$. Since $\mathcal{F}$ is a Cauchy filter, by Definition 2.5.2, for any $U \in \mathcal{F}(o)$ there exists $M \in \mathcal{F}$ s.t. $M - M \subset U$. But $\mathcal{F}'$ is finer than $\mathcal{F}$, so M belongs also to $\mathcal{F}'$. Hence, $\mathcal{F}'$ is obviously a Cauchy filter.
c) If a filter $\mathcal{F}$ converges to a point $x \in X$ then $\mathcal{F}(x) \subseteq \mathcal{F}$ (see Definition 1.1.27). By a), $\mathcal{F}(x)$ is a Cauchy filter and so b) implies that $\mathcal{F}$ itself is a Cauchy filter.

□

The converse of c) is in general false, in other words not every Cauchy filter converges.

Definition 2.5.5. *A subset A of a t.v.s. X is said to be* complete *if every Cauchy filter on A converges to a point x of A.*

It is important to distinguish between completeness and sequentially completeness.

Definition 2.5.6. *A subset A of a t.v.s. X is said to be* sequentially complete *if any Cauchy sequence in A converges to a point in A.*

It is easy to see that complete always implies sequentially complete. The converse is in general false (see Example 2.5.9). We will encounter an important class of t.v.s., the so-called metrizable spaces, for which the two notions coincide.

Proposition 2.5.7. *If a subset A of a t.v.s. X is complete then A is sequentially complete.*

Proof.
Let $S := \{x_n\}_{n\in\mathbb{N}}$ a Cauchy sequence of points in A. Then Proposition 2.5.3 guarantees that the filter $\mathcal{F}_S$ associated to S is a Cauchy filter in A. By the completeness of A we get that there exists $x \in A$ such that $\mathcal{F}_S$ converges to x. This is equivalent to say that the sequence S is convergent to $x \in A$ (see Proposition 1.1.29). Hence, A is sequentially complete. $\square$

Before showing an example of a subset of a t.v.s. which is sequentially complete but not complete, let us introduce two useful properties about completeness in t.v.s..

Proposition 2.5.8.
a) In a Hausdorff t.v.s. X, any complete subset is closed.
b) In a complete t.v.s. X, any closed subset is complete.

Example 2.5.9.
Let $X := \mathbb{R}^d$ where $d > \aleph_0$ endowed with the product topology given by considering each copy of $\mathbb{R}$ equipped with the usual topology given by the modulus. For convenience we write $X = \prod_{i\in J}\mathbb{R}$ with $|J| = d > \aleph_0$. Note that X is a Hausdorff t.v.s. as it is product of Hausdorff t.v.s.. Denote by H the subset of X consisting of all vectors $\underline{x} = (x_i)_{i\in J}$ in X with only countably many non-zero coordinates x_i. Claim: H is sequentially complete but not complete.

Proof. of Claim. Let us first make some observations on H.

- H is strictly contained in X.
  Indeed, any vector $\underline{y} \in X$ with all non-zero coordinates does not belong to H because $d > \aleph_0$.
- H is dense in X.
  In fact, let $\underline{x} = (x_i)_{i\in J} \in X$ and U a neighbourhood of $\underline{x}$ in X. Then, by definition of product topology on X, there exist $\prod_{i\in J} U_i \subseteq U$ s.t. $U_i \subseteq \mathbb{R}$ neighbourhood of x_i in $\mathbb{R}$ for all $i \in J$ and $U_i \neq \mathbb{R}$ for all $i \in I$ where $I \subset J$ with $|I| < \infty$. Take $\underline{y} := (y_i)_{i\in J}$ s.t. $y_i \in U_i$ for all $i \in J$ with $y_i \neq 0$ for all $i \in I$ and $y_i = 0$ otherwise. Then clearly $\underline{y} \in U$ but also $\underline{y} \in H$ because it has only finitely many non-zero coordinates. Hence, $U \cap H \neq \emptyset$ and so $\overline{H} = X$.

Now suppose that H is complete, then by Proposition 2.5.8-a) we have that H is closed. Therefore, by the density of H in X, it follows that $H = \overline{H} = X$ which contradicts the first of the property above. Hence, H is not complete.

In the end, let us show that H is sequentially complete. Let $(\underline{x}_n)_{n\in\mathbb{N}}$ a Cauchy sequence of vectors $\underline{x}_n = (x_n^{(i)})_{i\in J}$ in H. Then for each $i \in J$ we have that the sequence of the $i-th$ coordinates $(x_n^{(i)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. By the completeness (i.e. the sequentially completeness) of $\mathbb{R}$ we have that for each $i \in J$, the sequence $(x_n^{(i)})_{n\in\mathbb{N}}$ converges to a point $x^{(i)} \in \mathbb{R}$. Set $\underline{x} := (x^{(i)})_{i\in J}$. Then:

- $\underline{x} \in H$, because for each $n \in \mathbb{N}$ only countably many $x_n^{(i)} \neq 0$ and so only countably many $x^{(i)} \neq 0$.
- the sequence $(\underline{x}_n)_{n\in\mathbb{N}}$ converges to $\underline{x}$ in H. In fact, for any U neighbourhood of $\underline{x}$ in X there exist $\prod_{i\in J} U_i \subseteq U$ s.t. $U_i \in \mathbb{R}$ neighbourhood of x_i in $\mathbb{R}$ for all $i \in J$ and $U_i \neq \mathbb{R}$ for all $i \in I$ where $I \subset J$ with $|I| < \infty$. Since for each $i \in J$, the sequence $(x_n^{(i)})_{n\in\mathbb{N}}$ converges to $x^{(i)}$ in $\mathbb{R}$, we get that for each $i \in J$ there exists $N_i \in \mathbb{N}$ s.t. $x_n^{(i)} \in U_i$ for all $n \geq N_i$. Take $N := \max_{i\in I} N_i$ (the max exists because I is finite). Then for each $i \in J$ we get $x_n^{(i)} \in U_i$ for all $n \geq N$, i.e. $\underline{x}_n \in U$ for all $n \geq N$ which proves the convergence of $(\underline{x}_n)_{n\in\mathbb{N}}$ to $\underline{x}$.

Hence, we have showed that every Cauchy sequence in H is convergent. □

In order to prove Proposition 2.5.8, we need two small lemmas regarding convergence of filters in a topological space.

Lemma 2.5.10. *Let $\mathcal{F}$ be a filter of a topological Hausdorff space X. If $\mathcal{F}$ converges to $x \in X$ and also to $y \in X$, then $x = y$.*

Proof.
Suppose that $x \neq y$. Then, since X is Hausdorff, there exists $V \in \mathcal{F}(x)$ and $W \in \mathcal{F}(y)$ such that $V \cap W = \emptyset$. On the other hand, we know by assumption that $\mathcal{F} \to x$ and $\mathcal{F} \to y$ that is $\mathcal{F}(x) \subseteq \mathcal{F}$ and $\mathcal{F}(y) \subseteq \mathcal{F}$ (see Definition 1.1.27). Hence, $V, W \in \mathcal{F}$. Since filters are closed under finite intersections, we get that $V \cap W \in \mathcal{F}$ and so $\emptyset \in \mathcal{F}$ which contradicts the fact that $\mathcal{F}$ is a filter. □

Lemma 2.5.11. *Let A be a subset of a topological space X. Then $x \in \overline{A}$ if and only if there exists a filter $\mathcal{F}$ of subsets of X such that $A \in \mathcal{F}$ and $\mathcal{F}$ converges to x.*

Proof.
Let $x \in \overline{A}$, i.e. for any $U \in \mathcal{F}(x)$ in X we have $U \cap A \neq \emptyset$. Set $\mathcal{F} := \{F \subseteq X | U \cap A \subseteq F \text{ for some } U \in \mathcal{F}(x)\}$. It is easy to see that $\mathcal{F}$ is a filter of subsets of X. Therefore, for any $U \in \mathcal{F}(x)$, $U \cap A \in \mathcal{F}$ and $U \cap A \subseteq U$ imply that $U \in \mathcal{F}$, i.e. $\mathcal{F}(x) \subseteq \mathcal{F}$. Hence, $\mathcal{F} \to x$.

Viceversa, suppose that $\mathcal{F}$ is a filter of X s.t. $A \in \mathcal{F}$ and $\mathcal{F}$ converges to x. Let $U \in \mathcal{F}(x)$. Then $U \in \mathcal{F}$ since $\mathcal{F}(x) \subseteq \mathcal{F}$ by definition of convergence. Since also $A \in \mathcal{F}$ by assumption, we get $U \cap A \in \mathcal{F}$ and so $U \cap A \neq \emptyset$.

□

Proof. of Proposition 2.5.8

a) Let A be a complete subset of a Hausdorff t.v.s. X and let $x \in \overline{A}$. By Lemma 2.5.11, $x \in \overline{A}$ implies that there exists a filter $\mathcal{F}$ of subsets of X s.t. $A \in \mathcal{F}$ and $\mathcal{F}$ converges to x. Therefore, by Proposition 2.5.4-c), $\mathcal{F}$ is a Cauchy filter. Consider now $\mathcal{F}_A := \{U \in \mathcal{F} \,:\, U \subseteq A\} \subset \mathcal{F}$. It is easy to see that $\mathcal{F}_A$ is a Cauchy filter on A and so the completeness of A ensures that $\mathcal{F}_A$ converges to a point $y \in A$. Hence, any nbhood V of y in A belongs to $\mathcal{F}_A$ and so to $\mathcal{F}$. By definition of subset topology, this means that for any nbhood U of y in X we have $U \cap A \in \mathcal{F}$ and so $U \in \mathcal{F}$ (since $\mathcal{F}$ is a filter). Then $\mathcal{F}$ converges to y. Since X is Hausdorff, Lemma 2.5.10 establishes the uniqueness of the limit point of $\mathcal{F}$, i.e. $x = y$ and so $\overline{A} = A$.

b) Let A be a closed subset of a complete t.v.s. X and let $\mathcal{F}_A$ be any Cauchy filter on A. Take the filter $\mathcal{F} := \{F \subseteq X | B \subseteq F \text{ for some } B \in \mathcal{F}_A\}$. It is clear that $\mathcal{F}$ contains A and is finer than the Cauchy filter $\mathcal{F}_A$. Therefore, by Proposition 2.5.4-b), $\mathcal{F}$ is also a Cauchy filter. Then the completeness of the t.v.s. X gives that $\mathcal{F}$ converges to a point $x \in X$, i.e. $\mathcal{F}(x) \subseteq \mathcal{F}$. By Lemma 2.5.11, this implies that actually $x \in \overline{A}$ and, since A is closed, that $x \in A$. Now any neighbourhood of $x \in A$ in the subset topology is of the form $U \cap A$ with $U \in \mathcal{F}(x)$. Since $\mathcal{F}(x) \subseteq \mathcal{F}$ and $A \in \mathcal{F}$, we have $U \cap A \in \mathcal{F}$. Therefore, there exists $B \in \mathcal{F}_A$ s.t. $B \subseteq U \cap A \subset A$ and so $U \cap A \in \mathcal{F}_A$. Hence, $\mathcal{F}_A$ converges $x \in A$, i.e. A is complete. □

When a t.v.s. is not complete, it makes sense to ask if it is possible to embed it in a complete one. We are going to describe an abstract procedure that allows to always associate to an arbitrary Hausdorff t.v.s. X a complete Hausdorff t.v.s. $\hat{X}$ called the *completion* of X. Before doing that, we need to introduce uniformly continuous functions between t.v.s. and state some of their fundamental properties.

Definition 2.5.12. *Let X and Y be two t.v.s. and let A be a subset of X. A mapping $f : A \to Y$ is said to be* uniformly continuous *if for every neighborhood V of the origin in Y, there exists a neighborhood U of the origin in X such that for all pairs of elements $x_1, x_2 \in A$*

$$x_1 - x_2 \in U \;\Rightarrow\; f(x_1) - f(x_2) \in V.$$

Proposition 2.5.13. *Let X and Y be two t.v.s. and let A be a subset of X.*

a) If $f : A \to Y$ is uniformly continuous, then the image under f of a Cauchy filter on A is a Cauchy filter on Y.

b) If A is a linear subspace of X, then every continuous linear map from A to Y is uniformly continuous.

Proof. (Sheet 6, Exercise 2) □

Theorem 2.5.14.
Let X and Y be two Hausdorff t.v.s., A a dense subset of X, and $f : A \to Y$ a uniformly continuous mapping. If Y is complete the the following hold.
a) There exists a unique continuous mapping $\bar{f} : X \to Y$ which extends f, i.e. such that for all $x \in A$ we have $\bar{f}(x) = f(x)$.
b) $\bar{f}$ is uniformly continuous.
c) If we additionally assume that f is linear and A is a linear subspace of X, then $\bar{f}$ is linear.

Proof. (Sheet 6, Exercise 3) □

Let us now state and prove the theorem on completion of a t.v.s..

Theorem 2.5.15.
Let X be a Haudorff t.v.s.. Then there exists a complete Hausdorff t.v.s. $\hat{X}$ and a mapping $i : X \to \hat{X}$ with the following properties:
a) The mapping i is a topological monomorphism.
b) The image of X under i is dense in $\hat{X}$.
c) For every complete Hausdorff t.v.s. Y and for every continuous linear map $f : X \to Y$, there is a continuous linear map $\hat{f} : \hat{X} \to Y$ such that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"] & Y \\ \hat{X} \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

Furthermore:
I) Any other pair $(\hat{X}_1, i_1)$, consisting of a complete Hausdorff t.v.s. $\hat{X}_1$ and of a mapping $i_1 : X \to \hat{X}_1$ such that properties (a) and (b) hold substituting $\hat{X}$ with $\hat{X}_1$ and i with i_1, is topologically isomorphic to $(\hat{X}, i)$. This means that there is a topological isomorphism j of $\hat{X}$ onto $\hat{X}_1$ such that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "i"] & \hat{X}_1 \\ \hat{X} \arrow[ur, "j"'] & \end{tikzcd}$$

II) Given Y and f as in property (c), the continuous linear map $\hat{f}$ is unique.

Proof.

1) The set $\hat{X}$

Define the following relation on the collection of all Cauchy filters (c.f.) on X:

$$\mathcal{F} \sim_R \mathcal{G} \Leftrightarrow \forall\, U \,\text{nbhood of the origin in } X,\, \exists\, A \in \mathcal{F},\, \exists\, B \in \mathcal{G} \text{ s.t. } A - B \subset U.$$

The relation (R) is actually an equivalence relation. In fact:

- reflexive: If $\mathcal{F}$ is a c.f. on X, then by Definition 2.5.2 we have that for any U nbhood of the origin in X there exists $A \in \mathcal{F}$ s.t. $A - A \subset U$, i.e. $\mathcal{F} \sim_R \mathcal{F}$.
- symmetric: If $\mathcal{F}$ and $\mathcal{G}$ are c.f. on X s.t. $\mathcal{F} \sim_R \mathcal{G}$, then by definition of (R) we have that for any U nbhood of the origin in X there exist $A \in \mathcal{F}$ and $B \in \mathcal{G}$ s.t. $A - B \subset U$. This implies that $B - A \subset -U$, which gives $\mathcal{G} \sim_R \mathcal{F}$ considering that $-U$ is a generic nbhood of the origin in X in the same right as U.
- transitive: Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be c.f. on X s.t. $\mathcal{F} \sim_R \mathcal{G}$ and $\mathcal{G} \sim_R \mathcal{H}$. Take any U nbhood of the origin in X, then Theorem 2.1.10 ensures that there exists V nbhood of the origin in X s.t. $V + V \subset U$. By definition of (R), there exists $A \in \mathcal{F}$, $B_1, B_2 \in \mathcal{G}$ and $C \in \mathcal{H}$ s.t. $A - B_1 \subset V$ and $B_2 - C \subset V$. This clearly implies $A - (B_1 \cap B_2) \subset V$ and $(B_1 \cap B_2) - C \subset V$. By adding we obtain

$$A - C \subset A - (B_1 \cap B_2) + (B_1 \cap B_2) - C \subset V + V \subset U.$$

We define $\hat{X}$ as the quotient of the set of all c.f. on X w.r.t. the equivalence relation (R). Hence, an element $\hat{x}$ of $\hat{X}$ is an equivalence class of c.f. on X w.r.t. (R).

2) Operations on $\hat{X}$

Multiplication by a scalar

Let $0 \neq \lambda \in \mathbb{K}$ and let $\hat{x}$ be a generic element of $\hat{X}$. For any $\mathcal{F}$ any representative of $\hat{x}$, we define $\lambda\hat{x}$ to be the equivalence class w.r.t. (R) of the filter $\lambda\mathcal{F} := \{\lambda A \,:\, A \in \mathcal{F}\}$, i.e.

$$\lambda\hat{x} := \{\mathcal{G}\, \text{c.f. on}\, X \,:\, \mathcal{G} \sim_R \lambda\mathcal{F}\}.$$

It is easy to check that this definition does not depend on the choice of the representative $\mathcal{F}$ of $\hat{x}$ (see Sheet 7, Exercise 1).
When $\lambda = 0$, we have $\lambda\hat{x} = \hat{o}$, where $\hat{o}$ is the equivalence class w.r.t. (R) of the filter of neighborhoods of the origin o in X (or, which is the same, of the Cauchy filter consisting of all the subsets of X which contain o).

Vector addition

Let $\hat{x}$ and $\hat{y}$ be two arbitrary elements of $\hat{X}$, and $\mathcal{F}$ (resp. $\mathcal{G}$) a representative of $\hat{x}$ (resp. $\hat{y}$). We define $\hat{x}+\hat{y}$ to be the equivalence class w.r.t. (R) of the the filter $\mathcal{F}+\mathcal{G} := \{C \subseteq X \,:\, A+B \subseteq C \text{ for some } A \in \mathcal{F},\, B \in \mathcal{G}\}$, i.e.

$$\hat{x}+\hat{y} := \{\mathcal{H} \text{ c.f. on } X \,:\, \mathcal{H} \sim_R \mathcal{F}+\mathcal{G}\}.$$

Note that this vector addition is well-defined because its definition does not depend on the choice of the representative $\mathcal{F}$ of $\hat{x}$ and $\mathcal{G}$ of $\hat{y}$ (see Sheet 7, Exercise 1).

3) Topology on $\hat{X}$

Let U be an arbitrary nbhood of the origin in X. Define

$$\hat{U} := \{\hat{x} \in \hat{X} : U \in \mathcal{F} \text{ for some } \mathcal{F} \in \hat{x}\}. \tag{2.3}$$

and consider the collection $\hat{\mathcal{B}} := \{\hat{U} : U \text{ nbhood of the origin in } X\}$. The filter generated by $\hat{\mathcal{B}}$ fulfills all the properties in Theorem 2.1.10 (see Sheet 7, Exercise 2) and therefore, there exists a unique topology on $\hat{X}$ compatible with the vector space structure defined in Step 2 s.t. this is its filter of nbhoods of the origin $\hat{o} \in \hat{X}$. Clearly, $\hat{\mathcal{B}}$ is a basis of nbhoods of the origin $\hat{o} \in \hat{X}$ w.r.t. to such a topology.

4) $\hat{X}$ is a Hausdorff t.v.s.

So far we have constructed a t.v.s. $\hat{X}$. In this step, we aim to prove that $\hat{X}$ is also Hausdorff. By Proposition 2.2.3, it is enough to show that for any $\hat{x} \in \hat{X}$ with $\hat{o} \neq \hat{x}$ there exists a nbhood $\hat{V}$ of the origin $\hat{o}$ in $\hat{X}$ s.t. $\hat{x} \notin \hat{V}$.

Since $\hat{o} \neq \hat{x}$, for any $\mathcal{F} \in \hat{x}$ and for any $\mathcal{F}_o \in \hat{o}$ we have $\mathcal{F} \not\sim_R \mathcal{F}_o$. Take $\mathcal{F}_0 := \{E \subseteq X : o \in E\}$, then the fact that $\mathcal{F} \not\sim_R \mathcal{F}_o$ means that there exists U nbhood of the origin in X s.t. $\forall\, A \in \mathcal{F}$ and $\forall\, A_o \in \mathcal{F}_o$ we have $A - A_o \not\subset U$. In particular, $\{o\} \in \mathcal{F}_o$ and so $\forall\, A \in \mathcal{F}$ we get $A \not\subset U$, which simply means that $U \notin \mathcal{F}$. By Theorem 2.1.10 applied to the t.v.s. X, we can always find another nbohood V of the origin in X s.t. $V+V \subset U$.

Claim: V does not belong to any representative of $\hat{x}$. This means, in view of the definition (2.3), that $\hat{x} \notin \hat{V}$. Hence, as observed at the beginning, the conclusion follows by Proposition 2.2.3.

Let us finally prove the claim. If $\mathcal{F}'$ is any representative of $\hat{x}$, then $\mathcal{F} \sim_R \mathcal{F}'$, i.e. $\exists\, A \in \mathcal{F}$ and $\exists\, A' \in \mathcal{F}'$ s.t. $A - A' \subset V$. Suppose that $V \in \mathcal{F}'$ then $A' \cap V \in \mathcal{F}'$ and so $A' \cap V \neq \emptyset$. Therefore, we clearly have $A - (A' \cap V) \subset V$ which implies

$$A \subset V + (A' \cap V) \subset V + V \subset U.$$

Since $A \in \mathcal{F}$, this proves that $U \in \mathcal{F}$ which is a contradiction. Then $V \notin \mathcal{F}'$ for all $\mathcal{F}' \in \hat{x}$ that is exactly our claim.

5) Existence of $i : X \to \hat{X}$

We define the image of a point $x \in X$ under the mapping $i : X \to \hat{X}$ to be the equivalence class w.r.t. (R) of the filter $\mathcal{F}(x)$ of neighborhoods of x in X, i.e.

$$\forall\, x \in X,\;\; i(x) := \{\mathcal{F} \text{ c.f. on } X : \mathcal{F} \sim_R \mathcal{F}(x)\}.$$

Note that the following properties hold.

Lemma 2.5.16.

a) Two c.f. filters on X converging to the same point are equivalent w.r.t. (R)

b) If two c.f. filters $\mathcal{F}$ and $\mathcal{F}'$ on X are s.t. $\mathcal{F} \sim_R \mathcal{F}'$ and $\mathcal{F}'$ converges to $x \in X$ then also $\mathcal{F}$ converges to x.

Proof. (Sheet 7, Exercise 3) □

The previous lemma clearly proves that

$$i(x) \equiv \{\mathcal{F} \text{ c.f. on } X : \mathcal{F} \to x)\}.$$

6) i is a topological monomorphism (i.e. (a) holds)

i is injective

(see Sheet 7, Exercise 4).

i is a topological homomorphism

For the linearity of i see Sheet 7, Exercise 4. We aim to show here that i is both open and continuous on X.

To prove that i is open, we need to show that for any nbhood U of the origin in X the image $i(U)$ is a nbhood of the origin in $i(X)$ endowed with the subset topology induced by the topology on $\hat{X}$. Therefore, it suffices to show that for any nbhood U of the origin in X there exists U_1 nbhood of the origin in X s.t.

$$\hat{U_1} \cap i(X) \subseteq i(U) \tag{2.4}$$

where $\hat{U_1}$ is defined as in (2.3).

To show the continuity of i, we need to prove that for any nbhood $\hat{V}$ of the origin in $i(X)$ the preimage $i^{-1}(\hat{V})$ is a nbhood of the origin in X. Now any nbhood of the origin in $i(X)$ is of the form $\hat{U_1} \cap i(X)$ for some U_1 nbhood of the origin in X. Therefore, it is enough to show that for any U_1

nbhood of the origin in X there exists another U nbhood of the origin in X s.t. $U \subseteq i^{-1}(\hat{U}_1 \cap i(X))$ i.e.

$$i(U) \subseteq \hat{U}_1 \cap i(X) \tag{2.5}$$

In order to prove (2.4) and (2.5), we shall prove the following:

$$i(\mathring{V}) \subseteq \hat{V} \cap i(X) \subseteq i(\overline{V}), \quad \forall V \text{ nbhood of the origin in } X. \tag{2.6}$$

Indeed, if (2.6) holds then the first inclusion immediately shows (2.5) (for any U_1 nbhood of the origin in X take $U := \mathring{U}_1$ and apply the first inclusion of (2.6) to $V = U_1$). Moreover, (2.4) follows by combining the fact that for any nbhood U of the origin in X exists another nbhood U_1 of the origin in X s.t. $\overline{U_1} = U_1 \subseteq U$ (c.f. Sheet 3, Ex3-a)) together with the second inclusion of (2.6) (applied to U_1).

It remains to prove that (2.6) holds. Let V be any nbhood of the origin in X, then for any $x \in \mathring{V}$ we clearly have that V is a nbhood of x, which means that V belongs to a representative of $i(x)$, i.e. $i(x) \in \hat{V}$. Hence, $i(\mathring{V}) \subseteq \hat{V} \cap i(X)$. Now take $\hat{y} \in \hat{V} \cap i(X)$, i.e. $\hat{y} = i(x)$ for some $x \in X$ s.t. $i(x) \in \hat{V}$. Then, by definition (2.3), we have that $V \in \mathcal{F}$ for some $\mathcal{F} \in i(x)$ or in other words that V belongs to some filter $\mathcal{F}$ converging to x. Let W be another nbhood of the origin in X then $W + x$ is a nbhood of x in X and so $W + x \in \mathcal{F}$ (since $\mathcal{F} \to x$). Hence, $V \cap (W + x) \in \mathcal{F}$ which implies that $V \cap (W + x) \neq \emptyset$ i.e. $x \in \overline{V}$. This means that $\hat{y} = i(x) \in i(\overline{V})$ which proves $\hat{V} \cap i(X) \subseteq i(\overline{V})$.

7) $\overline{i(X)} = \hat{X}$ (i.e. (b) holds)

Let $\hat{x_o} \in \hat{X}$ and let N be any nbhood of $\hat{x_o}$ in $\hat{X}$. It suffices to consider the neighborhoods N of the form $\hat{U} + \hat{x_0}$ where $\hat{U}$ is defined by (2.3) for some U nbhood of the origin in X. We aim to prove that $(\hat{U} + \hat{x_o}) \cap i(X) \neq \emptyset$.

By Theorem 2.1.10, we know that for any U nbhood of the origin in X there exists V nbhood of the origin in X s.t. $V + V \subset U$. Let $\mathcal{F}_o$ be a representative of $\hat{x_0}$, then $\mathcal{F}_o$ is a c.f. on X and so there exists $A_o \in \mathcal{F}_o$ s.t. $A_o - A_o \subset V$. Fix an element $x \in A_o$. Then we get:

$$(V + x) - A_o \subset V + A_o - A_o \subset V + V \subset U. \tag{2.7}$$

Since $V + x$ is a nbhood of x in X, $V + x$ belongs to any Cauchy filter $\mathcal{F}$ converging to x and so $V + x \in \mathcal{F}$ for any $\mathcal{F} \in i(x)$. Then $(V + x) - A_o \in \mathcal{F} - \mathcal{F}_o$ and so (2.7) gives $U \in \mathcal{F} - \mathcal{F}_o$ i.e. $i(x) - \hat{x_o} \in \hat{U}$. Hence, we found that there exists $x \in X$ s.t. $i(x) \in \hat{U} + \hat{x_o}$ which gives the conclusion.

8) $\hat{X}$ is complete
Let $\hat{\mathcal{F}}$ be a Cauchy filter on $\hat{X}$. We aim to prove that there exists an element $\hat{x} \in \hat{X}$ s.t. $\hat{\mathcal{F}} \to \hat{x}$.

Consider the filter

$$\hat{\mathcal{F}}' := \{\hat{G} \subset \hat{X} : \hat{M}+\hat{U} \subset \hat{G} \text{ for some } \hat{M} \in \hat{\mathcal{F}} \text{ and } \hat{U} \text{ nbhood of the origin in } \hat{X}\}.$$

Note that $\hat{\mathcal{F}}' \subset \hat{\mathcal{F}}$ and $\hat{\mathcal{F}}'$ is also a Cauchy filter on $\hat{X}$. In fact, since $\hat{X}$ is a t.v.s., for any $\hat{U}$ nbhood of the origin in $\hat{X}$ there exists $\hat{V}_0$ balanced nbhood of the origin in $\hat{X}$ s.t. $\hat{V}_0 + \hat{V}_0 + \hat{V}_0 \subset U$. Take $\hat{V} := \frac{1}{3}\hat{V}_0$ which is also a nbhood of the origin in $\hat{X}$, then

$$\hat{V} + \hat{V} - \hat{V} \subset \hat{V}_0 + \hat{V}_0 + \hat{V}_0 \subset U.$$

Since $\hat{F}$ is a Cauchy filter, there exists $\hat{M} \in \hat{F}$ s.t. $\hat{M} - \hat{M} \subset \hat{V}$. Then

$$(\hat{M} + \hat{V}) - (\hat{M} + \hat{V}) \subset \hat{V} + \hat{V} - \hat{V} \subset U$$

Now let us consider the family of subsets of $i(X)$ given by

$$\mathcal{F}' := \{\hat{A} \cap i(X) : \hat{A} \in \hat{F}'\}.$$

It is possible to prove that $\mathcal{F}'$ is a filter on $i(X)$ and actually a Cauchy filter (see Sheet 7, Exercise 5). Moreover, since we proved that i is a topological isomorphism between X and $i(X)$, we have that $i^{-1}(\mathcal{F}')$ is a Cauchy filter on X. Take

$$\hat{x} := \{\mathcal{F} \text{ c.f. on } X : \mathcal{F} \sim_R i^{-1}(\mathcal{F}')\}.$$

Then $\hat{F}$ converges to $\hat{x}$ (see Sheet 7, Exercise 5).

9) Proof of the universal property (i.e. (c) and (II))
We can now identify X with $i(X)$ and so regard X as a dense linear subspace of $\hat{X}$. Since $f : X \to Y$ is continuous and linear by assumption, it is also uniformly continuous by Proposition 2.5.13. Then applying Theorem 2.5.14 with X replaced by $\hat{X}$ and A by X we get both the properties (c) and (II).

10) Uniqueness of $\hat{X}$ up to isomorphism (proof of (I))
Since by assumption $\hat{X}_1$ is a complete Hausdorff t.v.s. and $i_1 : X \to \hat{X}_1$ is a topological monomorphism (in particular i_1 is a continuous linear mapping), we have by (c) that there exists a unique continuous linear map $\hat{i_1}$ s.t. $\hat{i_1}(i(x)) = i_1(x)$ for any $x \in X$. Let us define $j := \hat{i_1}$. On the other hand, let us define $f : i_1(X) \to \hat{X}$ by $f(i_1(x)) = i(x)$ for any $x \in X$. Since f is

clearly linear and continuous and $i_1(X)$ is a linear subspace of $\hat{X}$, f is uniformly continuous and so by Theorem 2.5.14 we get that there exists a unique $\hat{f} : \hat{X_1} \to \hat{X}$ continuous and linear s.t. $\hat{f}(i_1(x)) = f(i_1(x))$ for any $x \in X$. Using the density of $i(X)$ in $\hat{X}$, the density of $i_1(X)$ in $\hat{X_1}$ and the continuity of the mappings involved, it is easy to check that

$$\hat{f}(j(\hat{x})) = \hat{x} \ \forall \, \hat{x} \in \hat{X}$$

and that

$$j(\hat{f}(\hat{x_1})) = \hat{x_1} \ \forall \, , \hat{x_1} \in \hat{X_1}.$$

This means that j and f are the inverse of each other and that both are topological isomorphisms.

□

Chapter 3

Finite dimensional topological vector spaces

3.1 Finite dimensional Hausdorff t.v.s.

Let X be a vector space over the field $\mathbb{K}$ of real or complex numbers. We know from linear algebra that the (algebraic) dimension of X, denoted by $\dim(X)$, is the cardinality of a basis of X. If $\dim(X)$ is finite, we say that X is *finite dimensional* otherwise X is *infinite dimensional*. In this section we are going to focus on finite dimensional vector spaces.

Let $\{e_1, \ldots, e_d\}$ be a basis of X, i.e. $\dim(X) = d$. Given any vector $x \in X$ there exist unique $x_1, \ldots, x_d \in \mathbb{K}$ s.t. $x = x_1e_1 + \cdots x_de_d$. This can be precisely expressed by saying that the mapping

$$\begin{array}{ccc} \mathbb{K}^d & \to & X \\ (x_1, \ldots, x_d) & \mapsto & x_1e_1 + \cdots x_de_d \end{array}$$

is an algebraic isomorphism (i.e. linear and bijective) between X and $\mathbb{K}^d$. In other words: *If X is a finite dimensional vector space then X is algebraically isomorphic to $\mathbb{K}^{\dim(X)}$.*

If now we give to X the t.v.s. structure and we consider $\mathbb{K}$ endowed with the euclidean topology, then it is natural to ask if such an algebraic isomorphism is by any chance a topological one, i.e. if it preserves the t.v.s. structure. The following theorem shows that if X is a finite dimensional Hausdorff t.v.s. then the answer is yes: X is topologically isomorphic to $\mathbb{K}^{\dim(X)}$. It is worth to observe that usually in applications we deal always with Hausdorff t.v.s., therefore it makes sense to mainly focus on them.

Theorem 3.1.1. *Let X be a finite dimensional Hausdorff t.v.s. over $\mathbb{K}$ (where $\mathbb{K}$ is endowed with the euclidean topology). Then:*
a) X is topologically isomorphic to $\mathbb{K}^d$, where $d = \dim(X)$.

b) Every linear functional on X is continuous.
c) Every linear map of X into any t.v.s. Y is continuous.

Before proving the theorem let us recall some lemmas about the continuity of linear functionals on t.v.s..

Lemma 3.1.2.
Let X be a t.v.s over $\mathbb{K}$ and $v \in X$. Then the following mapping is continuous.

$$\begin{array}{cccc} \varphi_v : & \mathbb{K} & \to & X \\ & \xi & \mapsto & \xi v. \end{array}$$

Proof. For any $\xi \in \mathbb{K}$, we have $\varphi_v(\xi) = M(\psi_v(\xi))$, where $\psi_v : \mathbb{K} \to \mathbb{K} \times X$ given by $\psi(\xi) := (\xi, v)$ is clearly continuous by definition of product topology and $M : \mathbb{K} \times X \to X$ is the scalar multiplication in the t.v.s. X which is continuous by definition of t.v.s.. Hence, φ_v is continuous as composition of continuous mappings. □

Lemma 3.1.3. *Let X be a t.v.s. over $\mathbb{K}$ and L a linear functional on X. Assume $L(x) \neq 0$ for some $x \in X$. Then the following are equivalent:*
a) L is continuous.
b) The null space $Ker(L)$ is closed in X
c) $Ker(L)$ is not dense in X.
d) L is bounded in some neighbourhood of the origin in X.

Proof. (see Sheet 4, Exercise 4)

Proof. of Theorem 3.1.1
Let $\{e_1, \ldots, e_d\}$ be a basis of X and let us consider the mapping

$$\begin{array}{cccc} \varphi : & \mathbb{K}^d & \to & X \\ & (x_1, \ldots, x_d) & \mapsto & x_1 e_1 + \cdots x_d e_d. \end{array}$$

As noted above, this is an algebraic isomorphism. Therefore, to conclude a) it remains to prove that φ is also a homeomorphism.

Step 1: φ is continuous.
When $d = 1$, we simply have $\varphi \equiv \varphi_{e_1}$ and so we are done by Lemma 3.1.2. When $d > 1$, for any $(x_1, \ldots, x_d) \in \mathbb{K}^d$ we can write: $\varphi(x_1, \ldots, x_d) = A(\varphi_{e_1}(x_1), \ldots, \varphi_{e_d}(x_d)) = A((\varphi_{e_1} \times \cdots \times \varphi_{e_d})(x_1, \ldots, x_d))$ where each φ_{e_j} is defined as above and $A : X \times X \to X$ is the vector addition in the t.v.s. X. Hence, φ is continuous as composition of continuous mappings.

Step 2: φ is open and b) holds.
We prove this step by induction on the dimension $\dim(X)$ of X.
For $\dim(X) = 1$, it is easy to see that φ is open, i.e. that the inverse of φ:

$$\begin{array}{rccc} \varphi^{-1}: & X & \to & \mathbb{K} \\ & x = \xi e_1 & \mapsto & \xi \end{array}$$

is continuous. Indeed, we have that

$$Ker(\varphi^{-1}) = \{x \in X : \varphi^{-1}(x) = 0\} = \{\xi e_1 \in X : \xi = 0\} = \{o\},$$

which is closed in X, since X is Hausdorff. Hence, by Lemma 3.1.3, φ^{-1} is continuous. This implies that b) holds. In fact, if L is a non-identically zero functional on X (when $L \equiv 0$, there is nothing to prove), then there exists a $o \neq \tilde{x} \in X$ s.t. $L(\tilde{x}) \neq 0$. W.l.o.g. we can assume $L(\tilde{x}) = 1$. Now for any $x \in X$, since $\dim(X) = 1$, we have that $x = \xi\tilde{x}$ for some $\xi \in \mathbb{K}$ and so $L(x) = \xi L(\tilde{x}) = \xi$. Hence, $L \equiv \varphi^{-1}$ which we proved to be continuous.

Suppose now that both a) and b) hold for $\dim(X) \leq d-1$. Let us first show that b) holds when $\dim(X) = d$. Let L be a non-identically zero functional on X (when $L \equiv 0$, there is nothing to prove), then there exists a $o \neq \tilde{x} \in X$ s.t. $L(\tilde{x}) \neq 0$. W.l.o.g. we can assume $L(\tilde{x}) = 1$. Note that for any $x \in X$ the element $x - \tilde{x}L(x) \in Ker(L)$. Therefore, if we take the canonical mapping $\phi : X \to X/Ker(L)$ then $\phi(x) = \phi(\tilde{x}L(x)) = L(x)\phi(\tilde{x})$ for any $x \in X$. This means that $X/Ker(L) = span\{\phi(\tilde{x})\}$ i.e. $\dim(X/Ker(L)) = 1$. Hence, $\dim(Ker(L)) = d-1$ and so by inductive assumption $Ker(L)$ is topologically isomorphic to $\mathbb{K}^{d-1}$. This implies that $Ker(L)$ is a complete subspace of X. Then, by Proposition 2.5.8-a), $Ker(L)$ is closed in X and so by Lemma 3.1.3 we get L is continuous. By induction, we can cocnlude that b) holds for any dimension $d \in \mathbb{N}$.

This immediately implies that a) holds for any dimension $d \in \mathbb{N}$. In fact, we just need to show that for any dimension $d \in \mathbb{N}$ the mapping

$$\begin{array}{rccc} \varphi^{-1}: & X & \to & \mathbb{K}^d \\ & x = \sum_{j=1}^{d} x_j e_j & \mapsto & (x_1, \ldots, x_d) \end{array}$$

is continuous. Now for any $x = \sum_{j=1}^{d} x_j e_j \in X$ we can write $\varphi^{-1}(x) = (L_1(x), \ldots, L_d(x))$, where for any $j \in \{1, \ldots, d\}$ we define $L_j : X \to \mathbb{K}$ by $L_j(x) := x_j e_j$. Since b) holds for any dimension, we know that each L_j is continuous and so φ^{-1} is continuous.

Step 3: The statement c) holds.
Let $f : X \to Y$ be linear and $\{e_1, \ldots, e_d\}$ be a basis of X. For any $j \in \{1, \ldots, d\}$ we define $b_j := f(e_j) \in Y$. Hence, for any $x = \sum_{j=1}^{d} x_j e_j \in X$ we

have $f(x) = f(\sum_{j=1}^{d} x_j e_j) = \sum_{j=1}^{d} x_j b_j$. We can rewrite f as composition of continuous maps i.e. $f(x) = A((\varphi_{b_1} \times \ldots \times \varphi_{b_d})(\varphi^{-1}(x))$ where:

- φ^{-1} is continuous by a)
- each φ_{b_j} is continuous by Lemma 3.1.2
- A is the vector addition on X and so it is continuous since X is a t.v.s..

Hence, f is continuous. □

Corollary 3.1.4 (Tychonoff theorem)**.** *Let $d \in \mathbb{N}$. The only topology that makes $\mathbb{K}^d$ a Hausdorff t.v.s. is the euclidean topology. Equivalently, on a finite dimensional vector space there is a unique topology that makes it into a Hausdorff t.v.s..*

Proof. We already know that $\mathbb{K}^d$ endowed with the euclidean topology τ_e is a Hausdorff t.v.s. of dimension d. Let us consider another topology τ on $\mathbb{K}^d$ s.t. $(\mathbb{K}^d, \tau)$ is also Hausdorff t.v.s.. Then Theorem 3.1.1-a) ensures that the identity map between $(\mathbb{K}^d, \tau_e)$ and $(\mathbb{K}^d, \tau)$ is a topological isomorphism. Hence, as observed at the end of Section 1.1.4 p.12, we get that $\tau \equiv \tau_e$. □

Corollary 3.1.5. *Every finite dimensional Hausdorff t.v.s. is complete.*

Proof. Let X be a Hausdorff t.v.s with $\dim(X) = d < \infty$. Then, by Theorem 3.1.1-a), X is topologically isomorphic to $\mathbb{K}^d$ endowed with the euclidean topology. Since the latter is a complete Hausdorff t.v.s., so is X. □

Corollary 3.1.6. *Every finite dimensional linear subspace of a Hausdorff t.v.s. is closed.*

Proof. Let S be a linear subspace of a Hausdorff t.v.s. (X, τ) and assume that $\dim(S) = d < \infty$. Then S endowed with the subspace topology induced by τ is itself a Hausdorff t.v.s. (see Sheet 5, Exercise 2). Hence, by Corollary 3.1.5 S is complete and therefore closed by Proposition 2.5.8-a). □

3.2 Connection between local compactness and finite dimensionality

By the Heine-Borel property (a subset of $\mathbb{K}^d$ is closed and bounded iff it is compact), $\mathbb{K}^d$ has a basis of compact neighborhoods of the origin (i.e. the closed balls centered at the origin in $\mathbb{K}^d$). Thus the origin, and consequently every point, of a finite dimensional Hausdorff t.v.s. has a basis of neighborhoods consisting of compact subsets. This means that *a finite dimensional Hausdorff* *t.v.s. is always locally compact.* Actually also the converse is true and gives the following beautiful characterization of finite dimensional Hausdorff t.v.s due to F. Riesz.

Theorem 3.2.1. *A Hausdorff t.v.s. is locally compact if and only if it is finite dimensional.*

For convenience let us recall the notions of compactness and local compactness for topological spaces before proving the theorem.

Definition 3.2.2. *A topological space X is* compact *if every open covering of X contains a finite subcovering. i.e. for any arbitrary collection $\{U_i\}_{i\in I}$ of open subsets of X s.t. $X \subseteq \cup_{i\in I} U_i$ there exists a finite subset J of I s.t. $X \subseteq \cup_{i\in J} U_i$.*

Definition 3.2.3. *A topological space X is* locally compact *if every point of X has a base of compact neighbourhoods.*

Just a small side remark: It is possible to show that every compact Hausdorff space is also locally compact but there exist locally compact spaces that are not compact such as:

- $\mathbb{K}^d$ with the euclidean topology
- any infinite set endowed with the discrete topology.
  Indeed, any set X with the discrete topology is locally compact, because for any $x \in X$ a basis of neighbourhoods of x in the discrete topology is given just $\{x\}$ which is open and also compact. However, if X is infinite then it is not compact. In fact, if we take the infinite open covering $\mathcal{S}$ of X given by all the singletons of its points, then any finite subcovering of $\mathcal{S}$ will not contain at least one point of X.

Proof. of Theorem 3.2.1
Let X be a locally compact Hausdorff t.v.s., and K a compact neighborhood of o in X. As K is compact and as $\frac{1}{2}K$ is a neighborhood of the origin (see Theorem 2.1.10-3), there is a finite family of points $x_1, \ldots, x_r \in X$ s.t.

$$K \subseteq \bigcup_{i=1}^{r} (x_i + \frac{1}{2}K).$$

Let $M := span\{x_1, \ldots, x_r\}$. Then M is a linear subspace of X and $\dim(M) < \infty$ is finite, hence M is closed in X by Corollary 3.1.6. Therefore, the quotient space X/M is Hausdorff t.v.s. by Proposition 2.3.5.

Let $\phi : X \to X/M$ be the canonical mapping. As $K \subseteq M + \frac{1}{2}K$, we have $\phi(K) \subseteq \phi(M) + \phi(\frac{1}{2}K) = \frac{1}{2}\phi(K)$, i.e. $2\phi(K) \subseteq \phi(K)$. By iterating we get

$\phi(2^n K) \subseteq \phi(K)$ for any $n \in \mathbb{N}$. As K is absorbing (see Theorem 2.1.10-5), we have $X = \bigcup_{n=1}^{\infty} 2^n K$. Thus

$$X/M = \phi(X) = \bigcup_{n=1}^{\infty} \phi(2^n K) \subseteq \phi(K).$$

Since ϕ is continuous and the continuous image of a compact set is compact, we get that $\phi(K)$ is compact. Thus X/M is a Hausdorff t.v.s. which is compact. We claim that X/M must be of zero dimension, i.e. reduced to one point. This concludes the proof because it implies $\dim(X) = \dim(M) < \infty$.

Let us prove the claim by contradiction. Suppose $\dim(X/M) > 0$ then X/M contains a subset of the form $\mathbb{R}\bar{x}$ for some $\bar{o} \neq \bar{x} \in X/M$. Since such a subset is closed and X/M is compact, $\mathbb{R}\bar{x}$ is also compact which is a contradiction. $\square$

Chapter 4

Locally convex topological vector spaces

4.1 Definition by neighbourhoods

Let us start this section by briefly recalling some basic properties of convex subsets of a vector space over $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$).

Definition 4.1.1. *A subset S of a vector space X over $\mathbb{K}$ is* convex *if, whenever S contains two points x and y, S also contains the segment of straight line joining them, i.e.*

$$\forall\, x, y \in S,\ \forall\, \alpha, \beta \geq 0 \ s.t.\ \alpha + \beta = 1, \alpha x + \beta y \in S.$$

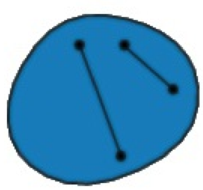

Figure 4.1: Convex set

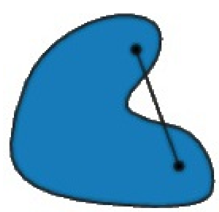

Figure 4.2: Not convex set

Examples 4.1.2.

a) The convex subsets of $\mathbb{R}$ are simply the intervals of $\mathbb{R}$. Examples of convex subsets of $\mathbb{R}^2$ are solid regular polygons. The Platonic solids are convex subsets of $\mathbb{R}^3$. Hyperplanes and halfspaces in $\mathbb{R}^n$ are convex.

b) Balls in a normed space are convex.

c) Consider a topological space X and the set $\mathcal{C}(X)$ of all real valued functions defined and continuous on X. $\mathcal{C}(X)$ with the pointwise addition and scalar *multiplication of functions is a vector space. Fixed* $g \in \mathcal{C}(X)$*, the subset* $S := \{f \in \mathcal{C}(X) : f(x) \geq g(x), \forall x \in X\}$ *is convex.*

d) *Consider the vector space* $\mathbb{R}[x]$ *of all polynomials in one variable with real coefficients. Fixed* $n \in \mathbb{N}$ *and* $c \in \mathbb{R}$*, the subset of all polynomials in* $\mathbb{R}[x]$ *such that the coefficient of the term of degree* n *is equal to* c *is convex.*

Proposition 4.1.3.
Let X *be a vector space. The following properties hold.*
- $\emptyset$ *and* X *are convex.*
- *Arbitrary intersections of convex sets are convex sets.*
- *Unions of convex sets are generally not convex.*
- *The sum of two convex sets is convex.*
- *The image and the preimage of a convex set under a linear map is convex.*

Definition 4.1.4. *Let* S *be any subset of a vector space* X*. We define the* convex hull *of* S*, denoted by* $conv(S)$*, to be the set of all finite convex linear combinations of elements of* S*, i.e.*

$$conv(S) := \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in S, \lambda_i \in [0,1], \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N} \right\}.$$

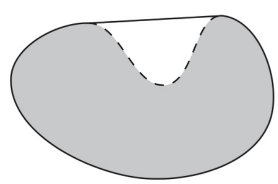

Figure 4.3: The solid line is the border of the convex hull of the shaded set

Proposition 4.1.5.
Let S, T *be arbitrary subsets of a vector space* X*. The following hold.*

a) $conv(S)$ *is convex*
b) $S \subseteq conv(S)$
c) *A set is convex if and only if it is equal to its own convex hull.*
d) *If* $S \subseteq T$ *then* $conv(S) \subseteq conv(T)$
e) $conv(conv(S)) = conv(S)$.
f) $conv(S + T) = conv(S) + conv(T)$.
g) *The convex hull of* S *is the smallest convex set containing* S*, i.e.* $conv(S)$ *is the intersection of all convex sets containing* S*.*
h) *The convex hull of a balanced set is balanced*

Proof. (Sheet 9, Exercise 1) □

Definition 4.1.6. *A subset S of a vector space X over $\mathbb{K}$ is* absolutely convex *if it is convex and balanced.*

Let us come back now to topological vector space.

Proposition 4.1.7. *The closure and the interior of convex sets in a t.v.s. are convex sets.*

Proof. Let S be a convex subset of a t.v.s. X. For any $\lambda \in [0,1]$, we define:

$$\begin{array}{rccc} \varphi_\lambda : & X \times X & \to & X \\ & (x,y) & \mapsto & \lambda x + (1-\lambda)y \end{array}.$$

Note that each φ_λ is continuous by the continuity of addition and scalar multiplication in the t.v.s. X. Since S is convex, for any $\lambda \in [0,1]$ we have that $\varphi_\lambda(S \times S) \subseteq S$ and so $\overline{\varphi_\lambda(S \times S)} \subseteq \overline{S}$. The continuity of φ_λ guarantees that $\varphi_\lambda(\overline{S \times S}) \subseteq \overline{\varphi_\lambda(S \times S)}$. Hence, we can conclude that $\varphi_\lambda(\overline{S} \times \overline{S}) = \varphi_\lambda(\overline{S \times S}) \subseteq \overline{S}$, i.e. $\overline{S}$ is convex.

To prove the convexity of the interior $\mathring{S}$, we must show that for any two points $x, y \in \mathring{S}$ and for any $\lambda \in [0,1]$ the point $z := \varphi_\lambda(x,y) \in \mathring{S}$.

By definition of interior points of S, there exists a neighborhood U of the origin in X such that $x + U \subseteq S$ and $y + U \subseteq S$. Then, of course, the claim is that $z + U \subseteq S$. This is indeed so, since for any element $u \in U$ we can write $z + u$ in the following form:

$$z + u = \lambda x + (1-\lambda)y + \lambda u + (1-\lambda)u = \lambda(x+u) + (1-\lambda)(y+u)$$

and since both vectors $x + u$ and $y + u$ belong to S, so does $z + u$. Hence, $z \in \mathring{S}$ which proves $\mathring{S}$ is convex. □

Definition 4.1.8. *A subset T of a t.v.s. is called a* barrel *if T has the following properties:*

1. *T is absorbing*
2. *T is absolutely convex*
3. *T is closed*

Proposition 4.1.9. *Every neighborhood of the origin in a t.v.s. is contained in a neighborhood of the origin which is a barrel.*

Proof.
Let U be a neighbourhood of the origin and define

$$T(U) := \overline{conv\left(\bigcup_{\lambda \in \mathbb{K}, |\lambda| \leq 1} \lambda U\right)}.$$

Clearly, $U \subseteq T(U)$. Therefore, $T(U)$ is a neighbourhood of the origin and so it is absorbing by Theorem 2.1.10. By construction, $T(U)$ is also closed and convex as closure of a convex set (see Proposition 4.1.7). To prove that $T(U)$ is a barrel it remains to show that it is balanced. It is easy to see that any point $z \in conv\left(\bigcup_{\lambda\in\mathbb{K},|\lambda|\leq 1}\lambda U\right)$ can be written as

$$z = tx + (1-t)y$$

with $0 \leq t \leq 1$, $x \in \lambda U$ and $y \in \mu U$, for some $\lambda, \mu \in \mathbb{K}$ s.t. $|\lambda| \leq 1$ and $|\mu| \leq 1$. Then for any $\xi \in \mathbb{K}$ with $|\xi| \leq 1$ we have:

$$\xi z = t(\xi x) + (1-t)(\xi y) \in conv\left(\bigcup_{\lambda\in\mathbb{K},|\lambda|\leq 1}\lambda U\right)$$

since $|\xi\lambda| \leq 1$ and $|\xi\mu| \leq 1$. This proves that $conv\left(\bigcup_{\lambda\in\mathbb{K},|\lambda|\leq 1}\lambda U\right)$ is balanced [1]. Hence, by Sheet 3 Exercise 2b, its closure $T(U)$ is also balanced. □

Corollary 4.1.10. *Every neighborhood of the origin in a t.v.s. is contained in a neighborhood of the origin which is absolutely convex.*

Note that the converse of Proposition 4.1.9 does not hold in any t.v.s.. Indeed, not every neighborhood of the origin contains another one which is a barrel. This means that not every t.v.s. has a basis of neighbourhood consisting of barrels. However, this is true for any locally convex t.v.s.

Definition 4.1.11. *A t.v.s. X is said to be* locally convex *(l.c.) if there is a basis of neighborhoods of the origin in X consisting of convex sets.*

Locally convex spaces are by far the most important class of t.v.s. and we will present later on several examples of such t.v.s.. For the moment let us focus on the properties of the filter of neighbourhoods of locally convex spaces.

Proposition 4.1.12. *A locally convex t.v.s. always has a basis of neighbourhoods of the origin consisting of open absorbing absolutely convex subsets.*

[1] One could also have directly observed that the set $\bigcup_{\lambda\in\mathbb{K},|\lambda|\leq 1}\lambda U$ is balanced and used Proposition 4.1.5-h to deduce that its convex hull is balanced.

Proof.
Let N be a neighbourhood of the origin in X. Since X is locally convex, there exists W convex neighbourhood of the origin in X s.t. $W \subseteq N$. Moreover, by Theorem 2.1.10, there exists U balanced neighbourhood of the origin in X s.t. $U \subseteq W$. The balancedness of U implies that $U = \bigcup_{\lambda\in\mathbb{K},|\lambda|\leq 1} \lambda U$. Then, using that W is a convex set containing U, we get

$$O := conv\left(\bigcup_{\lambda\in\mathbb{K},|\lambda|\leq 1} \lambda U\right) = conv(U) \subseteq W \subseteq N$$

and so $\mathring{O} \subseteq N$. Hence, the conclusion holds because $\mathring{O}$ is clearly open and convex and it is also balanced since $o \in \mathring{O}$ and O is balanced (as we have already showed in the proof of Proposition 4.1.9).

□

Similarly, we easily get that

Proposition 4.1.13. *A locally convex t.v.s. always has a basis of neighbourhoods of the origin consisting of barrels.*

Proof.
Let N be a neighbourhood of the origin in X. We know that every t.v.s. has a basis of closed neighbourhoods of the origin (see Sheet 3, Exercise 3a). Then there exists V closed neighbourhood of the origin in X s.t. $V \subseteq N$. Since X is locally convex, then there exists W convex neighbourhood of the origin in X s.t. $W \subseteq V$. Moreover, by Theorem 2.1.10, there exists U balanced neighbourhood of the origin in X s.t. $U \subseteq W$. Summing up we have: $U \subseteq W \subseteq V \subseteq N$ for some U, W, V neighbourhoods of the origin s.t. U balanced, W convex and V closed. The balancedness of U implies that $U = \bigcup_{\lambda\in\mathbb{K},|\lambda|\leq 1} \lambda U$. Then, using that W is a convex set containing U, we get

$$conv\left(\bigcup_{\lambda\in\mathbb{K},|\lambda|\leq 1} \lambda U\right) = conv(U) \subseteq W$$

Passing to the closures and using that V we get

$$T(U) = \overline{conv(U)} \subseteq \overline{W} \subseteq \overline{V} = V \subseteq N.$$

Hence, the conclusion holds because we have already showed in Proposition 4.1.9 that $T(U)$ is a barrel neighbourhood of the origin in X.

□

We can then characterize the class of locally convex t.v.s in terms of absorbing absolutely convex neighbourhoods of the origin.

Theorem 4.1.14. *If X is a l.c. t.v.s. then there exists a basis $\mathcal{B}$ of neighbourhoods of the origin consisting of absorbing absolutely convex subsets s.t.*
a) $\forall U, V \in \mathcal{B}$, $\exists W \in \mathcal{B}$ s.t. $W \subseteq U \cap V$
b) $\forall U \in \mathcal{B}$, $\forall \rho > 0$, $\exists W \in \mathcal{B}$ s.t. $W \subseteq \rho U$
Conversely, if $\mathcal{B}$ is a collection of absorbing absolutely convex subsets of a vector space X s.t. a) and b) hold, then there exists a unique topology compatible with the linear structure of X s.t. $\mathcal{B}$ is a basis of neighbourhoods of the origin in X for this topology (which is necessarily locally convex).

Proof. (Sheet 9, Exercise 2) □

In particular, the collection of all multiples ρU of an absorbing absolutely convex subset U of a vector space X is a basis of neighborhoods of the origin for a locally convex topology on X compatible with the linear structure (this ceases to be true, in general, if we relax the conditions on U).

4.2 Connection to seminorms

In applications it is often useful to define a locally convex space by means of a system of seminorms. In this section we will investigate the relation between locally convex t.v.s. and seminorms.

Definition 4.2.1. *Let X be a vector space. A function $p : X \to \mathbb{R}$ is called a* seminorm *if it satisfies the following conditions:*
1. *p is subadditive: $\forall x, y \in X$, $p(x + y) \leq p(x) + p(y)$.*
2. *p is positively homogeneous: $\forall x, y \in X$, $\forall \lambda \in \mathbb{K}, p(\lambda x) = |\lambda| p(x)$.*

Definition 4.2.2. *A seminorm p on a vector space X is a* norm *if $p^{-1}(\{0\}) = \{o\}$ (i.e. if $p(x) = 0$ implies $x = o$).*

Proposition 4.2.3. *Let p be a seminorm on a vector space X. Then the following properties hold:*
- *p is symmetric.*
- *$p(o) = 0$.*
- *$|p(x) - p(y)| \leq p(x - y)$, $\forall x, y \in X$.*
- *$p(x) \geq 0$, $\forall x \in X$.*
- *$Ker(p)$ is a linear subspace.*

Proof.

- The symmetry of p directly follows from the positive homogeneity of p. Indeed, for any $x \in X$ we have
$$p(-x) = p(-1 \cdot x) = |-1|p(x) = p(x).$$
- Using again the positive homogeneity of p we get that $p(o) = p(0 \cdot x) = 0 \cdot p(x) = 0$.
- For any $x, y \in X$, the subadditivity of p guarantees the following inequalities:
$$p(x) = p(x-y+y) \leq p(x-y)+p(y) \quad \text{and} \quad p(y) = p(y-x+x) \leq p(y-x)+p(x)$$
which establish the third property.
- The previous property directly gives the nonnegativity of p. In fact, for any $x \in X$ we get
$$0 \leq |p(x) - p(o)| \leq p(x - o) = p(x).$$
- Let $x, y \in Ker(p)$ and $\alpha, \beta \in \mathbb{K}$. Then
$$p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y) = 0$$
which implies $p(\alpha x + \beta y) = 0$, i.e. $\alpha x + \beta y \in Ker(p)$.

□

Examples 4.2.4.

a) *Suppose $X = \mathbb{R}^n$ and let M be a vector subspace of X. Set for any $x \in X$*
$$p_M(x) := \inf_{y \in M} \|x - y\|$$
where $\|\cdot\|$ is the Euclidean norm, i.e. $p_M(x)$ is the distance from the point x to M in the usual sense. If $dim(M) \geq 1$ then p_M is a seminorm and not a norm (M is exactly the kernel of p_M). When $M = \{o\}$, $p_M(\cdot) = \|\cdot\|$.

b) *Let X be a vector space on which is defined a nonnegative sesquilinear Hermitian form $B : X \times X \to \mathbb{K}$. Then the function*
$$p_B(x) := B(x, x)^{\frac{1}{2}}$$
is a seminorm. p_B is a norm if and only if B is positive definite (i.e. $B(x, x) > 0, \forall x \neq o$).

c) *Let $\mathcal{C}(\mathbb{R})$ be the vector space of all real valued continuous functions on the real line. For any bounded interval $[a,b]$ with $a,b \in \mathbb{R}$ and $a < b$, we define for any $f \in \mathcal{C}(\mathbb{R})$:*
$$p_{[a,b]}(f) := \sup_{a \leq t \leq b} |f(t)|.$$
$p_{[a,b]}$ is a seminorm but is never a norm because it might be that $f(t) = 0$ for all $t \in [a,b]$ (and so that $p_{[a,b]}(f) = 0$) but $f \not\equiv 0$. Other seminorms are the following ones:
$$q(f) := |f(0)| \quad \text{and} \quad q_p(f) := \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty.$$
Note that if $0 < p < 1$ then q_p is not subadditive and so it is not a seminorm.

Seminorms on vector spaces are strongly related to a special kind of functionals, i.e. *Minkowski functionals.* Let us investigate more in details such a relation. Note that we are still in the realm of vector spaces with no topology!

Definition 4.2.5. *Let X be a vector space and A a non-empty subset of X. We define the* Minkowski functional (or gauge) *of A to be the mapping:*

$$\begin{array}{rccl} p_A: & X & \to & \mathbb{R} \\ & x & \mapsto & p_A(x) := \inf\{\lambda > 0 : x \in \lambda A\} \end{array}$$

(where $p_A(x) = \infty$ if the set $\{\lambda > 0 : x \in \lambda A\}$ is empty).

It is then natural to ask whether there exists a class of subsets for which the associated Minkowski functionals are actually seminorms. The answer is positive for a class of subsets which we have already encountered in the previous section, namely for absorbing absolutely convex subsets. Actually we have even more as established in the following lemma.

Notation 4.2.6. *Let X be a vector space and p a seminorm on X. The sets*

$$\mathring{U}_p = \{x \in X : p(x) < 1\} \text{ and } U_p = \{x \in X : p(x) \leq 1\}.$$

are said to be, respectively, the closed and the open unit semiball *of p.*

Lemma 4.2.7. *Let X be a vector space. If A is a non-empty subset of X which is absorbing and absolutely convex, then the associated Minkowski functional p_A is a seminorm and $\mathring{U}_{p_A} \subseteq A \subseteq U_{p_A}$. Conversely, if q is a seminorm on X then $\mathring{U}_q$ is an absorbing absolutely convex set and $q = p_{\mathring{U}_q}$.*

Proof. Let A be a non-empty subset of X which is absorbing and absolutely convex and denote by p_A the associated Minkowski functional. We want to show that p_A is a seminorm.

- First of all, note that $p_A(x) < \infty$ for all $x \in X$ because A is absorbing. Indeed, by definition of absorbing set, for any $x \in X$ there exists $\rho_x > 0$ s.t. for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho_x$ we have $\lambda x \in A$ and so the set $\{\lambda > 0 : x \in \lambda A\}$ is never empty i.e. p_A has only finite nonnegative values. Moreover, since $o \in A$, we also have that $o \in \lambda A$ for any $\lambda \in \mathbb{K}$ and so $p_A(o) = \inf\{\lambda > 0 : o \in \lambda A\} = 0$.
- The balancedness of A implies that p_A is positively homogeneous. Since we have already showed that $p_A(o) = 0$ it remains to prove the positive homogeneity of p_A for non-null scalars. Since A is balanced we have that for any $x \in X$ and for any $\xi, \lambda \in \mathbb{K}$ with $\xi \neq 0$ the following holds:

$$\xi x \in \lambda A \text{ if and only if } x \in \frac{\lambda}{|\xi|} A. \tag{4.1}$$

  Indeed, A balanced guarantees that $\xi A = |\xi| A$ and so $x \in \frac{\lambda}{|\xi|} A$ is equivalent to $\xi x \in \lambda \frac{\xi}{|\xi|} A = \lambda A$. Using (4.1), we get that for any $x \in X$ and for any $\xi \in \mathbb{K}$ with $\xi \neq 0$:

$$\begin{aligned} p_A(\xi x) &= \inf\{\lambda > 0 : \xi x \in \lambda A\} \\ &= \inf\left\{\lambda > 0 : x \in \frac{\lambda}{|\xi|} A\right\} \\ &= \inf\left\{|\xi| \frac{\lambda}{|\xi|} > 0 : x \in \frac{\lambda}{|\xi|} A\right\} \\ &= |\xi| \inf\{\mu > 0 : x \in \mu A\} = |\xi| p_A(x). \end{aligned}$$

- The convexity of A ensures the subadditivity of p_A. Take $x, y \in X$. By definition of Minkowski functional, for every $\varepsilon > 0$ there exists $\lambda, \mu > 0$ s.t.

$$\lambda \leq p_A(x) + \varepsilon \text{ and } x \in \lambda A$$

  and

$$\mu \leq p_A(y) + \varepsilon \text{ and } y \in \mu A.$$

  Then, by the convexity of A, we obtain that $\frac{\lambda}{\lambda+\mu} A + \frac{\mu}{\lambda+\mu} A \subseteq A$, i.e. $\lambda A + \mu A \subseteq (\lambda + \mu) A$, and therefore $x + y \in (\lambda + \mu) A$. Hence:

$$p_A(x + y) = \inf\{\delta > 0 : x + y \in \delta A\} \leq \lambda + \mu \leq p_A(x) + p_A(y) + 2\varepsilon$$

  which proves the subadditivity of p_A since ε is arbitrary.

We can then conclude that p_A is a seminorm. Furthermore, we have the following inclusions:

$$\mathring{U}_{p_A} \subseteq A \subseteq U_{p_A}.$$

In fact, if $x \in \mathring{U}_{p_A}$ then $p_A(x) < 1$ and so there exists $0 \leq \lambda < 1$ s.t. $x \in \lambda A$. Since A is balanced, for such λ we have $\lambda A \subseteq A$ and therefore $x \in A$. On the other hand, if $x \in A$ then clearly $1 \in \{\lambda > 0 : x \in \lambda A\}$ which gives $p_A(x) \leq 1$ and so $x \in U_{p_A}$.

Conversely, let us take any seminorm q on X. Let us first show that $\mathring{U}_q$ is absorbing and absolutely convex and then that q coincides with the Minkowski functional associated to $\mathring{U}_q$.

- $\mathring{U}_q$ is absorbing.
  Let x be any point in X. If $q(x) = 0$ then clearly $x \in \mathring{U}_q$. If $q(x) > 0$, we can take $0 < \rho < \frac{1}{q(x)}$ and then for any $\lambda \in \mathbb{K}$ s.t. $|\lambda| \leq \rho$ the positive homogeneity of q implies that $q(\lambda x) = |\lambda| q(x) \leq \rho q(x) < 1$, i.e. $\lambda x \in \mathring{U}_q$.
- $\mathring{U}_q$ is balanced.
  For any $x \in \mathring{U}_q$ and for any $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$, again by the positive homogeneity of q, we get: $q(\lambda x) = |\lambda| q(x) \leq q(x) < 1$ i.e. $\lambda x \in \mathring{U}_q$.
- $\mathring{U}_q$ is convex.
  For any $x, y \in \mathring{U}_q$ and any $t \in [0,1]$, by both the properties of seminorm, we have that $q(tx + (1-t)y) \leq tq(x) + (1-t)q(y) < t + 1 - t = 1$ i.e. $tx + (1-t)y \in \mathring{U}_q$.

Moreover, for any $x \in X$ we easily see that

$$p_{\mathring{U}_q}(x) = \inf\{\lambda > 0 : x \in \lambda \mathring{U}_q\} = \inf\{\lambda > 0 : q(x) < \lambda\} = q(x). \qquad \square$$

We are now ready to see the connection between seminorms and locally convex t.v.s..

Definition 4.2.8. *Let X be a vector space and $\mathcal{P} := \{p_i\}_{i \in I}$ a family of seminorms on X. The coarsest topology $\tau_{\mathcal{P}}$ on X s.t. each p_i is continuous is said to be the* topology induced or generated by the family of seminorms $\mathcal{P}$*.*

Theorem 4.2.9. *Let X be a vector space and $\mathcal{P} := \{p_i\}_{i \in I}$ a family of seminorms. Then the topology induced by the family $\mathcal{P}$ is the unique topology making X into a locally convex t.v.s. and having as a basis of neighbourhoods of the origin in X the following collection:*

$$\mathcal{B} := \left\{ \{x \in X : p_{i_1}(x) < \varepsilon, \ldots, p_{i_n}(x) < \varepsilon\} : i_1, \ldots, i_n \in I, n \in \mathbb{N}, \varepsilon > 0, \epsilon \in \mathbb{R} \right\}.$$

Viceversa, the topology of an arbitrary locally convex t.v.s. is always induced by a family of seminorms (often called generating*).*

Proof. Let us first show that the collection $\mathcal{B}$ is a basis of neighbourhoods of the origin for the unique topology τ making X into a locally convex t.v.s. by using Theorem 4.1.14 and then let us prove that τ actually coincides with the topology induced by the family $\mathcal{P}$.

For any $i \in I$ and any $\varepsilon > 0$, consider the set $\{x \in X : p_i(x) < \varepsilon\} = \varepsilon \mathring{U}_{p_i}$. This is absorbing and absolutely convex, since we have already showed above that $\mathring{U}_{p_i}$ fulfills such properties. Therefore, any element of $\mathcal{B}$ is an absorbing absolutely convex subset of X as finite intersection of absorbing absolutely convex sets. Moreover, both properties a) and b) of Theorem 4.1.14 are clearly satisfied by $\mathcal{B}$. Hence, Theorem 4.1.14 guarantees that there exists a unique topology τ on X s.t. (X, τ) is a locally convex t.v.s. and $\mathcal{B}$ is a basis of neighbourhoods of the origin for τ.

Let us consider (X, τ). Then for any $i \in I$, the seminorm p_i is continuous, because for any $\varepsilon > 0$ we have $p_i^{-1}([0, \varepsilon[) = \{x \in X : p_i(x) < \varepsilon\} \in \mathcal{B}$ which means that $p_i^{-1}([0, \varepsilon[)$ is a neighbourhood of the origin in (X, τ). Therefore, the topology $\tau_{\mathcal{P}}$ induced by the family $\mathcal{P}$ is by definition coarser than τ. On the other hand, each p_i is also continuous w.r.t. $\tau_{\mathcal{P}}$ and so $\mathcal{B} \subseteq \tau_{\mathcal{P}}$. But $\mathcal{B}$ is a basis for τ, then necessarily τ is coarser than $\tau_{\mathcal{P}}$. Hence, $\tau \equiv \tau_{\mathcal{P}}$.

Viceversa, let us assume that (X, τ) is a locally convex t.v.s.. Then by Theorem 4.1.14 there exists a basis $\mathcal{N}$ of neighbourhoods of the origin in X consisting of absorbing absolutely convex sets s.t. the properties a) and b) in Theorem 4.1.14 are fulfilled. W.l.o.g. we can assume that they are open. Consider now the family $\mathcal{S} := \{p_N : N \in \mathcal{N}\}$. By Lemma 4.2.7, we know that each p_N is a seminorm and that $\mathring{U}_{p_N} \subseteq N$. Let us show that for any $N \in \mathcal{N}$ we have actually that $N = \mathring{U}_{p_N}$. Since any $N \in \mathcal{N}$ is open and the scalar multiplication is continuous we have that for any $x \in N$ there exists $0 < t < 1$ s.t. $x \in tN$ and so $p_N(x) \leq t < 1$, i.e. $x \in \mathring{U}_{p_N}$.

We want to show that the topology $\tau_{\mathcal{S}}$ induced by the family $\mathcal{S}$ coincides with original topology τ on X. We know from the first part of the proof how to construct a basis for a topology induced by a family of seminorms. In fact, a basis of neighbourhoods of the origin for $\tau_{\mathcal{S}}$ is given by

$$\mathcal{B} := \left\{ \bigcap_{i=1}^{n} \{x \in X : p_{N_i}(x) < \varepsilon\} : N_1, \ldots, N_n \in \mathcal{N}, n \in \mathbb{N}, \varepsilon > 0, \epsilon \in \mathbb{R} \right\}.$$

For any $N \in \mathcal{N}$ we have showed that $N = \mathring{U}_{p_N} \in \mathcal{B}$ so by Hausdorff criterion $\tau \subseteq \tau_{\mathcal{S}}$. Also for any $B \in \mathcal{B}$ we have $B = \cap_{i=1}^{n} \varepsilon \mathring{U}_{p_{N_i}} = \cap_{i=1}^{n} \varepsilon N_i$ for some $n \in \mathbb{N}$, $N_1, \ldots, N_n \in \mathcal{N}$ and $\varepsilon > 0$. Then the property b) (of Theorem 4.1.14) for $\mathcal{N}$ implies that for each $i = 1, \ldots, n$ there exists $V_i \in \mathcal{N}$ s.t. $V_i \subseteq \varepsilon N_i$ and so by the property a) of $\mathcal{N}$ we have that there exists $V \in \mathbb{N}$ s.t. $V \subseteq \cap_{i=1}^{n} V_i \subseteq B$. Hence, by Hausdorff criterion $\tau_{\mathcal{S}} \subseteq \tau$. $\square$

This result justifies why several authors define a locally convex space to be a t.v.s whose topology is induced by a family of seminorms (which is now evidently equivalent to Definition 4.1.11)

In the previous proofs we have used some interesting properties of semiballs in a vector space. For convenience, we collect them here together with some further ones which we will repeatedly use in the following.

Proposition 4.2.10. *Let X be a vector space and p a seminorm on X. Then:*

a) $\mathring{U}_p$ *is absorbing and absolutely convex.*

b) $\forall\, r > 0$, $r\mathring{U}_p = \{x \in X : p(x) < r\} = \mathring{U}_{\frac{1}{r}p}$.

c) $\forall\, x \in X$, $x + \mathring{U}_p = \{y \in X : p(y - x) < 1\}$.

d) If q is also a seminorm on X then: $p \leq q$ if and only if $\mathring{U}_q \subseteq \mathring{U}_p$.

e) If $n \in \mathbb{N}$ and $s_1, \ldots, s_n$ are seminorms on X, then their maximum *s defined as $s(x) := \max_{i=1,\ldots,n} s_i(x)$, $\forall x \in X$ is also seminorm on X and $\mathring{U}_s = \bigcap_{i=1}^{n} \mathring{U}_{s_i}$.*

All the previous properties also hold for closed semballs.

Proof.

a) This was already proved as part of Lemma 4.2.7.

b) For any $r > 0$, we have

$$r\mathring{U}_p = \{rx \in X : p(x) < 1\} = \underbrace{\{y \in X : \frac{1}{r}p(y) < 1\}}_{\mathring{U}_{\frac{1}{r}p}} = \{y \in X : p(y) < r\}.$$

c) For any $x \in X$, we have

$$x + \mathring{U}_p = \{x + z \in X : p(z) < 1\} = \{y \in X : p(y - x) < 1\}.$$

d) Suppose that $p \leq q$ and take any $x \in \mathring{U}_q$. Then we have $q(x) < 1$ and so $p(x) \leq q(x) < 1$, i.e. $x \in \mathring{U}_p$. Viceversa, suppose that $\mathring{U}_q \subseteq \mathring{U}_p$ holds and take any $x \in X$. We have that either $q(x) > 0$ or $q(x) = 0$. In the first case, for any $0 < \varepsilon < 1$ we get that $q\big(\frac{\varepsilon x}{q(x)}\big) = \varepsilon < 1$. Then $\frac{\varepsilon x}{q(x)} \in \mathring{U}_q$ which implies by our assumption that $\frac{\varepsilon x}{q(x)} \in \mathring{U}_p$ i.e. $p\big(\frac{\varepsilon x}{q(x)}\big) < 1$. Hence, $\varepsilon p(x) < q(x)$ and so when $\varepsilon \to 1$ we get $p(x) \leq q(x)$. If instead we are in the second case that is when $q(x) = 0$, then we claim that also $p(x) = 0$. Indeed, if $p(x) > 0$ then $q\big(\frac{x}{p(x)}\big) = 0$ and so $\frac{x}{p(x)} \in \mathring{U}_q$ which implies by our assumption that $\frac{x}{p(x)} \in \mathring{U}_p$, i.e. $p(x) < p(x)$ which is a contradiction.

e) It is easy to check, using basic properties of the maximum, that the subadditivity and the positive homogeneity of each s_i imply the same properties for s. In fact, for any $x, y \in X$ and for any $\lambda \in \mathbb{K}$ we get:

- $s(x+y) = \max_{i=1,\dots,n} s_i(x+y) \leq \max_{i=1,\dots,n} (s_i(x)+s_i(y))$
  $\leq \max_{i=1,\dots,n} s_i(x) + \max_{i=1,\dots,n} s_i(y) = s(x)+s(y)$
- $s(\lambda x) = \max_{i=1,\dots,n} s_i(\lambda x) = |\lambda| \max_{i=1,\dots,n} s_i(x) = |\lambda| s(x)$.

Moreover, if $x \in \mathring{U}_s$ then $\max_{i=1,\dots,n} s_i(x) < 1$ and so for all $i = 1,\dots,n$ we have $s_i(x) < 1$, i.e. $x \in \bigcap_{i=1}^n \mathring{U}_{s_i}$. Conversely, if $x \in \bigcap_{i=1}^n \mathring{U}_{s_i}$ then for all $i = 1,\dots,n$ we have $s_i(x) < 1$. Since $s(x)$ is the maximum over a finite number of terms, it will be equal to $s_j(x)$ for some $j \in \{1,\dots,n\}$ and therefore $s(x) = s_j(x) < 1$, i.e. $x \in \mathring{U}_s$.

□

Proposition 4.2.11. *Let X be a t.v.s. and p a seminorm on X. Then the following conditions are equivalent:*
a) the open unit semiball $\mathring{U}_p$ of p is an open set.
b) p is continuous at the origin.
c) the closed unit semiball U_p of p is a barrel neighbourhood of the origin.
d) p is continuous at every point.

Proof.
a) $\Rightarrow$ *b)* Suppose that $\mathring{U}_p$ is open in the topology on X. Then for any $\varepsilon > 0$ we have that $p^{-1}([0,\varepsilon[) = \{x \in X : p(x) < \varepsilon\} = \varepsilon\mathring{U}_p$ is an open neighbourhood of the origin in X. This is enough to conclude that $p : X \to \mathbb{R}^+$ is continuous at the origin.

b) $\Rightarrow$ *c)* Suppose that p is continuous at the origin, then $U_p = p^{-1}([0,1])$ is a closed neighbourhood of the origin. Since U_p is also absorbing and absolutely convex by Proposition 4.2.10-a), U_p is a barrel.

c) $\Rightarrow$ *d)* Assume that c) holds and fix $o \neq x \in X$. Using Proposition 4.2.10 and Proposition 4.2.3, we get that for any $\varepsilon > 0$: $p^{-1}([-\varepsilon + p(x), p(x)+\varepsilon]) = \{y \in X : |p(y) - p(x)| \leq \varepsilon\} \supseteq \{y \in X : p(y-x) \leq \varepsilon\} = x + \varepsilon U_p$, which is a closed neighbourhood of x since X is a t.v.s. and by the assumption *c)*. Hence, p is continuous at x.

d) $\Rightarrow$ *a)* If p is continuous on X then a) holds because the preimage of an open set under a continuous function is open and $\mathring{U}_p = p^{-1}([0,1[)$. □

With such properties in our hands we are able to give a criterion to compare two locally convex topologies using their generating families of seminorms.

Theorem 4.2.12 (Comparison of l.c. topologies)**.**
Let $\mathcal{P} = \{p_i\}_{i\in I}$ and $\mathcal{Q} = \{q_j\}_{j\in J}$ be two families of seminorms on the vector space X inducing respectively the topologies $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$, which both make X into a locally convex t.v.s.. Then $\tau_{\mathcal{P}}$ is finer than $\tau_{\mathcal{Q}}$ (i.e. $\tau_{\mathcal{Q}} \subseteq \tau_{\mathcal{P}}$) iff

$$\forall q \in \mathcal{Q}\ \exists n \in \mathbb{N},\, i_1, \dots, i_n \in I,\, C > 0 \ \text{ s.t. } \ Cq(x) \leq \max_{k=1,\dots,n} p_{i_k}(x),\ \forall x \in X. \tag{4.2}$$

Proof.
Let us first recall that, by Theorem 4.2.9, we have that

$$\mathcal{B}_{\mathcal{P}} := \left\{ \bigcap_{k=1}^{n} \varepsilon \mathring{U}_{p_{i_k}} : i_1, \dots, i_n \in I, n \in \mathbb{N}, \varepsilon > 0, \varepsilon \in \mathbb{R} \right\}$$

and

$$\mathcal{B}_{\mathcal{Q}} := \left\{ \bigcap_{k=1}^{n} \varepsilon \mathring{U}_{q_{j_k}} : j_1, \dots, j_n \in J, n \in \mathbb{N}, \varepsilon > 0, \varepsilon \in \mathbb{R} \right\}.$$

are respectively bases of neighbourhoods of the origin for $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$.

By using Proposition 4.2.10, the condition (4.2) can be rewritten as

$$\forall q \in \mathcal{Q},\ \exists n \in \mathbb{N},\, i_1, \dots, i_n \in I,\, C > 0 \ \text{ s.t. } \ C \bigcap_{k=1}^{n} \mathring{U}_{p_{i_k}} \subseteq \mathring{U}_q.$$

which means that

$$\forall q \in \mathcal{Q},\ \exists\ B_q \in \mathcal{B}_{\mathcal{P}} \ \text{ s.t. } \ B_q \subseteq \mathring{U}_q. \tag{4.3}$$

since $C \bigcap_{k=1}^{n} \mathring{U}_{p_{i_k}} \in \mathcal{B}_{\mathcal{P}}$.

Condition (4.3) means that for any $q \in \mathcal{Q}$ the set $\mathring{U}_q \in \tau_{\mathcal{P}}$, which by Proposition 4.2.11 is equivalent to say that q is continuous w.r.t. $\tau_{\mathcal{P}}$. By definition of $\tau_{\mathcal{Q}}$, this gives that $\tau_{\mathcal{Q}} \subseteq \tau_{\mathcal{P}}$. [2]

□

[2] **Alternate proof without using Prop 4.2.11.** (Sheet 10, Exercise 1)
Suppose that (4.2) holds and take any $B' \in \mathcal{B}_{\mathcal{Q}}$, i.e. $B' = \bigcap_{k=1}^{m} \varepsilon \mathring{U}_{q_{j_k}}$ for some $m \in \mathbb{N}$, $j_1, \dots, j_n \in J$ and $0 < \varepsilon \in \mathbb{R}$. Then, by using m times the condition (4.3), we obtain that there exist $B_1, \dots, B_m \in \mathcal{B}_{\mathcal{P}}$ such that $\forall k \in \{1, \dots, m\}$, $B_k \subseteq \mathring{U}_{q_{j_k}}$. Hence, $\bigcap_{k=1}^{m} B_i \subseteq \bigcap_{k=1}^{m} \mathring{U}_{q_{j_k}}$. Multiplying by ε both sides of the inclusion, we get $B' \supseteq \varepsilon \bigcap_{k=1}^{m} B_i \in \mathcal{B}_{\mathcal{P}}$ and so, by Hausdorff criterion (see Theorem 1.1.16) $\tau_{\mathcal{Q}} \subseteq \tau_P$.

Conversely, suppose that τ_P is finer than $\tau_{\mathcal{Q}}$ and take any $q \in \mathcal{Q}$. Since $\mathring{U}_q \in \mathcal{B}_{\mathcal{Q}}$, by Hausdorff criterion, we get that there exists $B \in \mathcal{B}_P$ s.t. $B \subseteq \mathring{U}_q$. Now such B will be of the form $B = \bigcap_{k=1}^{n} \varepsilon \mathring{U}_{p_{i_k}}$ for some $n \in \mathbb{N}, i_1, \dots, i_n \in I$ and $0 < \varepsilon \in \mathbb{R}$. Then, Proposition 4.2.10 gives that $B = \varepsilon \mathring{U}_{\max\limits_{k=1,\dots,n} p_{i_k}} \subseteq \mathring{U}_q$, i.e. $\mathring{U}_{\max\limits_{k=1,\dots,n} p_{i_k}} \subseteq \mathring{U}_{\varepsilon q}$ which is equivalent to $\varepsilon q(x) \leq \max\limits_{k=1,\dots,n} p_{i_k}(x)$, $\forall x \in X$.

This theorem allows us to easily see that the topology induced by a family of seminorms on a vector space does not change if we close the family under taking the maximum of finitely many of its elements. Indeed, the following result holds.

Proposition 4.2.13. *Let $\mathcal{P} := \{p_i\}_{i\in I}$ be a family of seminorms on a vector space X and $\mathcal{Q} := \left\{ \max_{i\in B} p_i : \emptyset \neq B \subseteq I \text{ with } B \text{ finite} \right\}$. Then $\mathcal{Q}$ is a family of seminorms and $\tau_{\mathcal{P}} = \tau_{\mathcal{Q}}$, where $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$ denote the topology induced on X by $\mathcal{P}$ and $\mathcal{Q}$, respectively.*

Proof.
First of all let us note that, by Proposition 4.2.10, $\mathcal{Q}$ is a family of seminorms. On the one hand, since $\mathcal{P} \subseteq \mathcal{Q}$, by definition of induced topology we have $\tau_{\mathcal{P}} \subseteq \tau_{\mathcal{Q}}$. On the other hand, for any $q \in \mathcal{Q}$ we have $q = \max_{i\in B} p_i$ for some $\emptyset \neq B \subseteq I$ finite. Then (4.2) is fulfilled for $n = |B|$ (where $|B|$ denotes the cardinality of the finite set B), $i_1, \ldots, i_n$ being the n elements of B and for any $0 < C \leq 1$. Hence, by Theorem 4.2.12, $\tau_{\mathcal{Q}} \subseteq \tau_{\mathcal{P}}$. $\square$

This fact can be used to show the following very useful property of locally convex t.v.s.

Proposition 4.2.14. *The topology of a locally convex t.v.s. can be always induced by a* directed *family of seminorms.*

Definition 4.2.15. *A family $\mathcal{Q} := \{q_j\}_{j\in J}$ of seminorms on a vector space X is said to be* directed *if*

$$\forall\, j_1, j_2 \in J,\, \exists\, j \in J, C > 0 \;\; s.t. \;\; Cq_j(x) \geq \max\{q_{j_1}(x), q_{j_2}(x)\},\, \forall\, x \in X \quad (4.4)$$

or equivalently by induction if

$$\forall\, n \in \mathbb{N}, j_1, \ldots, j_n \in J,\, \exists\, j \in J, C > 0 \;\; s.t. \;\; Cq_j(x) \geq \max_{k=1,\ldots,n} q_{j_k}(x),\, \forall\, x \in X.$$

Proof. of Proposition 4.2.14
Let (X, τ) be a locally convex t.v.s.. By Theorem 4.2.9, we have that there exists a family of seminorms $\mathcal{P} := \{p_i\}_{i\in I}$ on X s.t. $\tau = \tau_{\mathcal{P}}$. Let us define $\mathcal{Q}$ as the collection obtained by forming the maximum of finitely many elements of $\mathcal{P}$, i.e. $\mathcal{Q} := \left\{ \max_{i\in B} p_i : \emptyset \neq B \subseteq I \text{ with } B \text{ finite} \right\}$. By Proposition 4.2.13, $\mathcal{Q}$ is a family of seminorms and we have that $\tau_{\mathcal{P}} = \tau_{\mathcal{Q}}$. We claim that $\mathcal{Q}$ is directed.

Let $q, q' \in \mathcal{Q}$, i.e. $q := \max_{i \in B} p_i$ and $q' := \max_{i \in B'} p_i$ for some non-empty finite subsets B, B' of I. Let us define $q'' := \max_{i \in B \cup B'} p_i$. Then $q'' \in \mathcal{Q}$ and for any $C \geq 1$ we have that (4.4) is satisfied, because we get that for any $x \in X$

$$Cq''(x) = C \max \left\{ \max_{i \in B} p_i(x), \max_{i \in B'} p_i(x) \right\} \geq \max\{q(x), q'(x)\}.$$

Hence, $\mathcal{Q}$ is directed. $\square$

It is possible to show (Sheet 10, Exercise 2) that a basis of neighbourhoods of the origin for the l.c. topology $\tau_{\mathcal{Q}}$ induced by a directed family of seminorms $\mathcal{Q}$ is given by:

$$\mathcal{B}_d := \{r \mathring{U}_q : q \in \mathcal{Q}, r > 0\}. \tag{4.5}$$

4.3 Hausdorff locally convex t.v.s

In Section 2.2, we gave some characterization of Hausdorff t.v.s. which can of course be applied to establish whether a locally convex t.v.s. is Hausdorff or not. However, in this section we aim to provide necessary and sufficient conditions bearing only on the family of seminorms generating a locally convex topology for being a Hausdorff topology.

Definition 4.3.1.
A family of seminorms $\mathcal{P} := \{p_i\}_{i \in I}$ on a vector space X is said to be separating *if*

$$\forall x \in X \setminus \{o\}, \exists\ i \in I \ \text{ s.t. } \ p_i(x) \neq 0. \tag{4.6}$$

Note that the separation condition (4.6) is equivalent to

$$p_i(x) = 0, \, \forall i \in I \ \Rightarrow x = o$$

which by using Proposition 4.2.10 can be rewritten as

$$\bigcap_{i \in I, c > 0} c \mathring{U}_{p_i} = \{o\},$$

since $p_i(x) = 0$ is equivalent to say that $p_i(x) < c$, for all $c > 0$.

Lemma 4.3.2. *Let $\tau_{\mathcal{P}}$ be the topology induced by a separating family of seminorms $\mathcal{P} := (p_i)_{i\in I}$ on a vector space X. Then $\tau_{\mathcal{P}}$ is a Hausdorff topology.*

Proof. Let $x, y \in X$ be s.t. $x \neq y$. Since $\mathcal{P}$ is separating, $\exists\ i \in I$ s.t. $p_i(x-y) \neq 0$. Then $\exists\ \epsilon > 0$ s.t. $p_i(x-y) = 2\epsilon$. Let us define $V_x := \{u \in X \mid p_i(x-u) < \epsilon\}$ and $V_y := \{u \in X \mid p_i(y-u) < \epsilon\}$. By Proposition 4.2.10, we get that $V_x = x + \varepsilon \mathring{U}_{p_i}$ and $V_y = y + \varepsilon \mathring{U}_{p_i}$. Since Theorem 4.2.9 guarantees that $(X, \tau_{\mathcal{P}})$ is a t.v.s. where the set $\varepsilon \mathring{U}_{p_i}$ is a neighbourhood of the origin, V_x and V_y are neighbourhoods of x and y, respectively. They are clearly disjoint. Indeed, if there would exist $u \in V_x \cap V_y$ then

$$p_i(x-y) = p_i(x-u+u-y) \leq p_i(x-u) + p_i(u-y) < 2\varepsilon$$

which is a contradiction. □

Proposition 4.3.3. *A locally convex t.v.s. is Hausdorff if and only if its topology can be induced by a separating family of seminorms.*

Proof. Let (X, τ) be a locally convex t.v.s.. Then we know that there always exists a basis $\mathcal{N}$ of neighbourhoods of the origin in X consisting of open absorbing absolutely convex sets. Moreover, in Theorem 4.2.9, we have showed that $\tau = \tau_{\mathcal{P}}$ where $\mathcal{P}$ is the family of seminorms given by the Minkowski functionals of sets in $\mathcal{N}$, i.e. $\mathcal{P} := \{p_N : N \in \mathcal{N}\}$, and also that for each $N \in \mathcal{N}$ we have $N = \mathring{U}_{p_N}$.

Suppose that (X, τ) is also Hausdorff. Then Proposition 2.2.3 ensures that for any $x \in X$ with $x \neq o$ there exists a neighbourhood V of the origin in X s.t. $x \notin V$. This implies that there exists at least $N \in \mathcal{N}$ s.t. $x \notin N$ [3]. Hence, $x \notin N = \mathring{U}_{p_N}$ means that $p_N(x) \geq 1$ and so $p_N(x) \neq 0$, i.e. $\mathcal{P}$ is separating.

Conversely, if τ is induced by a separating family of seminorms $\mathcal{P}$, i.e. $\tau = \tau_{\mathcal{P}}$, then Lemma 4.3.2 ensures that X is Hausdorff.

□

Examples 4.3.4.

1. *Every normed space is a Hausdorff locally convex space, since every norm is a seminorm satisfying the separation property. Therefore, every Banach space is a complete Hausdorff locally convex space.*

[3] Since $\mathcal{N}$ is a basis of neighbourhoods of the origin, $\exists\ N \in \mathcal{N}$ s.t. $N \subseteq V$. If x would belong to all elements of the basis then in particular it would be $x \in N$ and so also $x \in V$, contradiction.

2. *Every family of seminorms on a vector space containing a norm induces a Hausdorff locally convex topology.*
3. *Given an open subset Ω of $\mathbb{R}^d$ with the euclidean topology, the space $\mathcal{C}(\Omega)$ of real valued continuous functions on Ω with the so-called* topology of uniform convergence on compact sets *is a locally convex t.v.s.. This topology is defined by the family $\mathcal{P}$ of all the seminorms on $\mathcal{C}(\Omega)$ given by*
$$p_K(f) := \max_{x\in K} |f(x)|, \ \forall K \subset \Omega \ \textit{compact} .$$
Moreover, $(\mathcal{C}(\Omega), \tau_{\mathcal{P}})$ is Hausdorff, because the family $\mathcal{P}$ is clearly separating. In fact, if $p_K(f) = 0$, $\forall K$ compact subsets of Ω then in particular $p_{\{x\}}(f) = |f(x)| = 0$ $\forall x \in \Omega$, which implies $f \equiv 0$ on Ω.
More generally, for any X locally compact we have that $\mathcal{C}(X)$ with the topology of uniform convergence on compact subsets of X is a locally convex Hausdorff t.v.s.

To introduce two other examples of l.c. Hausdorff t.v.s. we need to recall some standard general notations. Let $\mathbb{N}_0$ be the set of all non-negative integers. For any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ one defines $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. For any $\beta \in \mathbb{N}_0^d$, the symbol D^β denotes the partial derivative of order $|\beta|$ where $|\beta| := \sum_{i=1}^d \beta_i$, i.e.
$$D^\beta := \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}} = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}}.$$

Examples 4.3.5.

1. *Let $\Omega \subseteq \mathbb{R}^d$ open in the euclidean topology. For any $k \in \mathbb{N}_0$, let $\mathcal{C}^k(\Omega)$ be the set of all real valued $k-$*times continuously differentiable functions *on Ω, i.e. all the derivatives of f of order $\leq k$ exist (at every point of Ω) and are continuous functions in Ω. Clearly, when $k = 0$ we get the set $\mathcal{C}(\Omega)$ of all real valued continuous functions on Ω and when $k = \infty$ we get the so-called set of all* infinitely differentiable functions *or* smooth functions *on Ω. For any $k \in \mathbb{N}_0$, $\mathcal{C}^k(\Omega)$ (with pointwise addition and scalar multiplication) is a vector space over $\mathbb{R}$. The topology given by the following family of seminorms on $\mathcal{C}^k(\Omega)$:*
$$p_{m,K}(f) := \sup_{\substack{\beta\in\mathbb{N}_0^d \\ |\beta|\leq m}} \sup_{x\in K} \left|(D^\beta f)(x)\right|, \ \forall K \subseteq \Omega \ \textit{compact}, \forall m \in \{0, 1, \ldots, k\},$$
makes $\mathcal{C}^k(\Omega)$ into a locally convex Hausdorff t.v.s.. (Note that when $k = \infty$ we have $m \in \mathbb{N}_0$.)

2. *The* Schwartz space *or* space of rapidly decreasing functions *on* $\mathbb{R}^d$ *is defined as the set* $\mathcal{S}(\mathbb{R}^d)$ *of all real-valued functions which are defined and infinitely differentiable on* $\mathbb{R}^d$ *and which have the additional property (regulating their growth at infinity) that all their derivatives tend to zero at infinity faster than any inverse power of* x*, i.e.*

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} \left| x^\alpha D^\beta f(x) \right| < \infty, \ \forall \alpha, \beta \in \mathbb{N}_0^d \right\}.$$

(For example, any smooth function f *with compact support in* $\mathbb{R}^d$ *is in* $\mathcal{S}(\mathbb{R}^d)$*, since any derivative of* f *is continuous and supported on a compact subset of* $\mathbb{R}^d$*, so* $x^\alpha(D^\beta f(x))$ *has a maximum in* $\mathbb{R}^d$ *by the extreme value theorem.)*

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ *is a vector space over* $\mathbb{R}$ *and the topology given by the family* $\mathcal{Q}$ *of seminorms on* $\mathcal{S}(\mathbb{R}^d)$*:*

$$q_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^d} \left| x^\alpha D^\beta f(x) \right|, \ \forall \alpha, \beta \in \mathbb{N}_0^d$$

makes $\mathcal{S}(\mathbb{R}^d)$ *into a locally convex Hausdorff t.v.s.. Indeed, the family is clearly separating, because if* $q_{\alpha,\beta}(f) = 0$*,* $\forall \alpha, \beta \in \mathbb{N}_0^d$ *then in particular* $q_{o,o}(f) = \sup_{x\in\mathbb{R}^d} |f(x)| = 0 \ \forall x \in \mathbb{R}^d$*, which implies* $f \equiv 0$ *on* $\mathbb{R}^d$*.*
Note that $\mathcal{S}(\mathbb{R}^d)$ *is a linear subspace of* $\mathcal{C}^\infty(\mathbb{R}^d)$*, but its topology* $\tau_{\mathcal{Q}}$ *on* $\mathcal{S}(\mathbb{R}^d)$ *is finer than the subspace topology induced on it by* $\mathcal{C}^\infty(\mathbb{R}^d)$*. (Sheet 10, Exercise 1)*

4.4 The finest locally convex topology

In the previous sections we have seen how to generate topologies on a vector space which makes it into a locally convex t.v.s.. Among all of them, there is the finest one (i.e. the one having the largest number of open sets) which is usually called the *finest locally convex topology* on the given vector space.

Proposition 4.4.1. *The finest locally convex topology on a vector space* X *is the topology induced by the family of all seminorms on* X *and it is a Hausdorff topology.*

Proof.
Let us denote by $\mathcal{S}$ the family of all seminorms on the vector space X. By Theorem 4.2.9, we know that the topology $\tau_{\mathcal{S}}$ induced by $\mathcal{S}$ makes X into a locally convex t.v.s. We claim that $\tau_{\mathcal{S}}$ is the finest locally convex topology. In

fact, if there was a finer locally convex topology τ (i.e. if $\tau_{\mathcal{S}} \subseteq \tau$ with (X, τ) locally convex t.v.s.) then Theorem 4.2.9 would give that τ is also induced by a family $\mathcal{P}$ of seminorms. But surely $\mathcal{P} \subseteq \mathcal{S}$ and so $\tau = \tau_{\mathcal{P}} \subseteq \tau_{\mathcal{S}}$ by definition of induced topology. Hence, $\tau = \tau_{\mathcal{S}}$.

It remains to show that $(X, \tau_{\mathcal{S}})$ is Hausdorff. By Lemma 4.3.2, it is enough to prove that $\mathcal{S}$ is separating. Let $x \in X \setminus \{o\}$ and let $\mathcal{B}$ be an algebraic basis of the vector space X containing x. Define the linear functional $L : X \to \mathbb{R}$ as $L(x) = 1$ and $L(y) = 0$ for all $y \in \mathcal{B} \setminus \{x\}$. Then it is easy to see that $s := |L|$ is a seminorm, so $s \in \mathcal{S}$ and $s(x) \neq 0$, which proves that $\mathcal{S}$ is separating. $\square$

An alternative way of describing the finest locally convex topology on a vector space X without using the seminorms is the following:

Proposition 4.4.2. *The collection of all absorbing absolutely convex sets of a vector space X is a basis of neighbourhoods of the origin for the finest locally convex topology on X.*

Proof. Let τ_{max} be the finest locally convex topology on X and $\mathcal{A}$ the collection of all absorbing absolutely convex sets of X. By Theorem 4.1.14, we know that every locally convex t.v.s. has a basis of neighbourhood of the origin consisting of absorbing absolutely convex subsets of X. Then clearly the basis of neighbourhoods of the origin $\mathcal{B}_{max}$ of τ_{max} is contained in $\mathcal{A}$. Hence, $\tau_{max} \subseteq \tau$ where τ denote the topology generated by $\mathcal{A}$. On the other hand, $\mathcal{A}$ fulfills all the properties required in Theorem 4.1.14 and so τ also makes X into a locally convex t.v.s.. Hence, by definition of finest locally convex topology, $\tau \subseteq \tau_{max}$. $\square$

This result can be clearly proved also using the Proposition 4.4.1 and the correspondence between Minkowski functionals and absorbing absolutely convex subsets of X introduced in the Section 4.2.

Proposition 4.4.3. *Every linear functional on a vector space X is continuous w.r.t. the finest locally convex topology on X.*

Proof. Let $L : X \to \mathbb{K}$ be a linear functional on a vector space X. For any $\varepsilon > 0$, we denote by $B_\varepsilon(0)$ the open ball in $\mathbb{K}$ of radius ε and center $0 \in \mathbb{K}$, i.e. $B_\varepsilon(0) := \{k \in \mathbb{K} : |k| < \varepsilon\}$. Then we have that $L^{-1}(B_\varepsilon(0)) = \{x \in X : |L(x)| < \varepsilon\}$. It is easy to verify that the latter is an absorbing absolutely convex subset of X and so, by Proposition 4.4.2, it is a neighbourhood of the origin in the finest locally convex topology on X. Hence L is continuous at the origin and so, by Proposition 2.1.15-3), L is continuous everywhere in X. $\square$

4.5 Direct limit topology on a countable dimensional t.v.s.

In this section we are going to give an important example of finest locally convex topology on an infinite dimensional vector space, namely the *direct limit topology* on any countable dimensional vector space. For simplicity, we are going to focus on $\mathbb{R}-$vector spaces.

Definition 4.5.1. *Let X be an infinite dimensional vector space whose dimension is countable. The* direct limit topology *(or* finite topology*) τ_f on X is defined as follows:*
$U \subseteq X$ is open in τ_f iff $U \cap W$ is open in the euclidean topology on W, $\forall\, W \subset X$ with $dim(W) < \infty$.
Equivalently, if we fix a Hamel basis $\{x_n\}_{n\in\mathbb{N}}$ of X and if for any $n \in \mathbb{N}$ we set $X_n := span\{x_1, \ldots, x_n\}$ s.t. $X = \bigcup_{i=1}^{\infty} X_i$ and $X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$, then $U \subseteq X$ is open in τ_f iff $U \cap X_i$ is open in the euclidean topology on V_i for every $i \in \mathbb{N}$.

Theorem 4.5.2. *Let X be an infinite dimensional vector space whose dimension is countable endowed with the finite topology τ_f. Then:*
a) (X, τ_f) is a Hausdorff locally convex t.v.s.
b) τ_f is the finest locally convex topology on X

Proof.
a) We leave to the reader the proof of the fact that τ_f is compatible with the linear structure of X (Sheet 10, Exercise 3) and we focus instead on proving that τ_f is a locally convex topology. To this aim we are going to show that for any open neighbourhood U of the origin in (X, τ_f) there exists an open convex neighbourhood $U' \subseteq U$.

Let $\{x_i\}_{i\in\mathbb{N}}$ be an $\mathbb{R}$-basis for X and set $X_n := span\{x_1, \ldots, x_n\}$ for any $n \in \mathbb{N}$. We proceed (by induction on $n \in \mathbb{N}$) to construct an increasing sequence $C_n \subseteq U \cap X_n$ of convex subsets as follows:

- For $n = 1$: Since $U \cap X_1$ is open in $X_1 = \mathbb{R}x_1$, we have that there exists $a_1 \in \mathbb{R}, a_1 > 0$ such that $C_1 := \left\{\lambda_1 x_1 \mid -a_1 \leq \lambda_1 \leq a_1\right\} \subseteq U \cap X_1$.
- Inductive assumption on n: We assume we have found $a_1, \ldots, a_n \in \mathbb{R}_+$ such that $C_n := \left\{\lambda_1 x_1 + \ldots + \lambda_n x_n \mid -a_i \leq \lambda_i \leq a_i\,; i \in \{1, \ldots, n\}\right\} \subseteq U \cap X_n$. Note that C_n is closed (in X_n, as well as) in X_{n+1}.
- For $n+1$: We claim $\exists\, a_{n+1} > 0, a_{n+1} \in \mathbb{R}$ such that
  $C_{n+1} := \left\{\lambda_1 x_1 + \ldots + \lambda_n x_n + \lambda_{n+1} x_{n+1} \middle| -a_i \leq \lambda_i \leq a_i\,; i \in \{1, \ldots, n+1\}\right\} \subseteq U \cap X_{n+1}$.
  Proof of claim: If the claim does not hold, then $\forall\, N \in \mathbb{N}\ \exists\, x^N \in X_{n+1}$ s.t.
  $$x^N = \lambda_1^N x_1 + \ldots \lambda_n^N x_n + \lambda_{n+1}^N x_{n+1}$$

with $-a_i \leq \lambda_i^N \leq a_i$ for $i \in \{1, \ldots, n\}$, $-\frac{1}{N} \leq \lambda_{n+1}^N \leq \frac{1}{N}$ and $x^N \notin U$.

But x^N has form $x^N = \underbrace{\lambda_1^N x_1 + \ldots + \lambda_n^N x_n}_{\in C_n} + \lambda_{n+1}^N x_{n+1}$, so $\{x^N\}_{N\in\mathbb{N}}$ is a bounded sequence in $X_{n+1} \setminus U$. Therefore, we can find a subsequence $\{x^{N_j}\}_{j\in\mathbb{N}}$ which is convergent as $j \to \infty$ to $x \in C_n \subseteq U$ (since C_n is closed in X_{n+1} and the $n+1-$th component of x^{N_j} tends to 0 as $j \to \infty$). Hence, the sequence $\{x^{N_j}\}_{j\in\mathbb{N}} \subseteq X_{n+1} \setminus U$ converges to $x \in U$ but this contradicts the fact that $X_{n+1} \setminus U$ is closed in X_{n+1}. This establishes the claim.

Now for any $n \in \mathbb{N}$ consider

$$D_n := \left\{ \lambda_1 x_1 + \ldots + \lambda_n x_n \mid -a_i < \lambda_i < a_i \; ; i \in \{1, \ldots, n\} \right\},$$

then $D_n \subset C_n \subseteq U \cap X_n$ is open and convex in X_n. Then $U' := \cup_{n\in\mathbb{N}} D_n$ is an open and convex neighbourhood of the origin in (X, τ_f) and $U' \subseteq U$.

b) Let us finally show that τ_f is actually the finest locally convex topology τ_{max} on X. Since we have already showed that τ_f is a l.c. topology on X, clearly we have $\tau_f \subseteq \tau_{max}$ by definition of finest l.c. topology on X.

Conversely, let us consider $U \subseteq X$ open in τ_{max}. We want to show that U is open in τ_f, i.e. $W \cap U$ is open in the euclidean topology on W for any finite dimensional subspace W of X. Now each W inherits τ_{max} from X. Let us denote by τ_{max}^W the subspace topology induced by (X, τ_{max}) on W. By definition of subspace topology, we have that $W \cap U$ is open in τ_{max}^W. Moreover, by Proposition 4.4.1, we know that (X, τ_{max}) is a Hausdorff t.v.s. and so (W, τ_{max}^W) is a finite dimensional Hausdorff t.v.s. (see by Proposition 2.1.15-1). Therefore, τ_{max}^W has to coincide with the euclidean topology by Theorem 3.1.1 and, consequently, $W \cap U$ is open w.r.t. the euclidean topology on W. □

We actually already know a concrete example of countable dimensional space with the finite topology:

Example 4.5.3. *Let $n \in \mathbb{N}$ and $\underline{x} = (x_1, \ldots, x_n)$. Denote by $\mathbb{R}[\underline{x}]$ the space of polynomials in the n variables $x_1, \ldots, x_n$ with real coefficients and by*

$$\mathbb{R}_d[\underline{x}] := \{f \in \mathbb{R}[\underline{x}] | \deg f \leq d\}, d \in \mathbb{N}_0,$$

then $\mathbb{R}[\underline{x}] := \bigcup_{d=0}^{\infty} \mathbb{R}_d[\underline{x}]$. The finite topology τ_f on $\mathbb{R}[\underline{x}]$ is then given by: $U \subseteq \mathbb{R}[\underline{x}]$ is open in τ_f iff $\forall d \in \mathbb{N}_0, U \cap \mathbb{R}_d[\underline{x}]$ is open in $\mathbb{R}_d[\underline{x}]$ with the euclidean topology.

4.6 Continuity of linear mappings on locally convex spaces

Since locally convex spaces are a particular class of topological vector spaces, the natural functions to be considered on this spaces are continuous linear maps. In this section, we present a necessary and sufficient condition for the continuity of a linear map between two l.c. spaces, bearing only on the seminorms inducing the two topologies.

For simplicity, let us start with linear functionals on a l.c. space. Recall that for us $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ endowed with the euclidean topology given by the absolute value $|\cdot|$. In this section, for any $\varepsilon > 0$ we denote by $B_\varepsilon(0)$ the open ball in $\mathbb{K}$ of radius ε and center $0 \in \mathbb{K}$ i.e. $B_\varepsilon(0) := \{k \in \mathbb{K} : |k| < \varepsilon\}$.

Proposition 4.6.1. *Let τ be a locally convex topology on a vector space X generated by a directed family $\mathcal{Q}$ of seminorms on X. Then $L : X \to \mathbb{K}$ is a τ-continuous linear functional iff there exists $q \in \mathcal{Q}$ such that L is q-continuous, i.e.*

$$\exists q \in \mathcal{Q}, \exists C > 0 \ \ s.t. \, |L(x)| \leq Cq(x), \, \forall \, x \in X. \tag{4.7}$$

Proof.
Let us first observe that since X and $\mathbb{K}$ are both t.v.s. by Proposition 2.1.15-3) the continuity of L is equivalent to its continuity at the origin. Therefore, it is enough to prove the criterion for the continuity of L at the origin.

($\Rightarrow$) Suppose that L is τ-continuous at the origin in X. Then we have that $L^{-1}(B_1(0)) = \{x \in X : |L(x)| < 1\}$ is an open neighbourhood of the origin in (X, τ). Since the family $\mathcal{Q}$ inducing τ is directed, a basis of neighbourhood of the origin in (X, τ) is given by $\mathcal{B}_d$ as in (4.5). Therefore, $\exists B \in \mathcal{B}_d$ s.t. $B \subseteq L^{-1}(B_1(0))$, i.e.

$$\exists \ q \in \mathcal{Q}, \ \exists \ r > 0 \ \text{ s.t. } \ r\mathring{U}_q \subseteq L^{-1}(B_1(0)). \tag{4.8}$$

Then for any $\varepsilon > 0$ we get $r\varepsilon\mathring{U}_q \subseteq \varepsilon L^{-1}(B_1(0)) = L^{-1}(B_\varepsilon(0))$. This proves that L is $q-$continuous at the origin, because $r\varepsilon\mathring{U}_q$ is clearly an open neighbourhood of the origin in X w.r.t. the topology generated by the single seminorm q.

($\Leftarrow$) Suppose that there exists $q \in \mathcal{Q}$ s.t. L is q-continuous in X. Then, since τ is the topology induced by the whole family $\mathcal{Q}$ which is finer than the one induced by the single seminorm q, we clearly have that L is also $\tau-$continuous.

Note that by simply observing that $|L|$ is a seminorm and by using Proposition 4.2.10 we get that (4.7) is equivalent to (4.8) and so to the q-continuity of L at the origin.

$\square$

By using this result together with Proposition 4.2.14 we get the following.

Corollary 4.6.2. *Let τ be a locally convex topology on a vector space X generated by a family $\mathcal{P} := \{p_i\}_{i\in I}$ of seminorms on X. Then $L : X \to \mathbb{K}$ is a τ-continuous linear functional iff there exist $n \in \mathbb{N}$, $i_1, \ldots, i_n \in I$ such that L is $\left(\max_{k=1,\ldots,n} p_{i_k}\right)$-continuous, i.e.*

$$\exists\, n \in \mathbb{N}, \exists\, i_1, \ldots, i_n \in I, \exists\, C > 0 \;\; s.t. \;\; |L(x)| \leq C \max_{k=1,\ldots,n} p_{i_k}(x), \, \forall\, x \in X.$$

The proof of Proposition 4.6.1 can be easily modified to get the following more general criterion for the continuity of any linear map between two locally convex spaces.

Theorem 4.6.3. *Let X and Y be two locally convex t.v.s. whose topologies are respectively generated by the families $\mathcal{P}$ and $\mathcal{Q}$ of seminorms on X. Then $f : X \to Y$ linear is continuous iff*

$$\forall\, q \in \mathcal{Q}, \exists\, n \in \mathbb{N}, \exists\, p_1, \ldots, p_n \in \mathcal{P}, \exists\, C > 0 \,:\, q(f(x)) \leq C \max_{i=1,\ldots,n} p_i(x), \, \forall\, x \in X.$$

Proof. (Exercise) □

Chapter 5

The Hahn-Banach Theorem and its applications

5.1 The Hahn-Banach Theorem

One of the most important results in the theory of t.v.s. is the Hahn-Banach theorem (HBT). It is named for Hans Hahn and Stefan Banach who proved this theorem independently in the late 1920s, dealing with the problem of extending continuous linear functionals defined on a subspace of a seminormed vector space to the whole space. We will see that actually this extension problem can be reduced to the problem of separating by a closed hyperplane a convex open set and an affine submanifold (the image by a translation of a linear subspace) which do not intersect. Indeed, there are two main versions of HBT showing respectively the analytic and the geometric side of this result.

Before stating these two versions of HBT, let us recall the notion of hyperplane in a vector space (we always consider vector spaces over the field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$). A *hyperplane* H in a vector space X over $\mathbb{K}$ is a maximal proper linear subspace of X or, equivalently, a linear subspace of codimension one, i.e. $dim X/H = 1$. Another equivalent formulation is that a hyperplane is a set of the form $\varphi^{-1}(\{0\})$ for some linear functional $\varphi : X \to \mathbb{K}$ not identically zero. The translation by a non-null vector of a hyperplane will be called *affine hyperplane*.

Theorem 5.1.1 (Analytic form of Hahn-Banach theorem).
Let p be a seminorm on a vector space X over $\mathbb{K}$, M a linear subspace of X, and f a linear functional on M such that

$$|f(x)| \leq p(x), \, \forall \, x \in M. \tag{5.1}$$

There exists a linear functional $\tilde{f}$ on X extending f, i.e. $\tilde{f}(x) = f(x) \, \forall \, x \in M$, and such that

$$|\tilde{f}(x)| \leq p(x), \, \forall \, x \in X. \tag{5.2}$$

Theorem 5.1.2 (Geometric form of Hahn-Banach theorem).
Let X be a topological vector space over $\mathbb{K}$, N a linear subspace of X, and Ω a convex open subset of X such that $N \cap \Omega = \emptyset$. Then there exists a closed hyperplane H of X such that

$$N \subseteq H \quad \textit{and} \quad H \cap \Omega = \emptyset. \tag{5.3}$$

It should be remarked that the vector space X does not apparently carry any topology in Theorem 5.1.1, but actually the datum of a seminorm on X is equivalent to the datum of the topology induced by this seminorm. It is then clear that the conditions (5.1) and (5.2) imply the $p-$continuity of the functions f and $\tilde{f}$, respectively.

Let us also stress the fact that in Theorem 5.1.2 neither local convexity nor the Hausdorff separation property are assumed on the t.v.s. X. Moreover, it is easy to see that the geometric form of HBT could have been stated also in an affine setting, namely starting with any affine submanifold N of X which does not intersect the open convex subset Ω and getting a closed affine hyperplane fulfilling (5.3).

We first prove Theorem 5.1.2 and then show how to derive from this the analytic form Theorem 5.1.1.

Before starting the proof, let us fix one more definition. A *cone* C in a vector space X over $\mathbb{R}$ is a subset of X which is closed under addition and multiplication by positive scalars.

Proof. Theorem 5.1.2
We assume that $\Omega \neq \emptyset$, otherwise there is nothing to prove.

1) Existence of a linear subspace H of X maximal for (5.3).
This first part of the proof is quite simple and consists in a straightforward application of Zorns lemma. In fact, consider the family $\mathcal{F}$ of all the linear subspaces L of X such that

$$N \subseteq L \quad \text{and} \quad L \cap \Omega = \emptyset. \tag{5.4}$$

$\mathcal{F}$ is clearly non-empty since N belongs to it by assumption. If we take now a totally ordered subfamily $\mathcal{C}$ of $\mathcal{F}$ (totally ordered for the inclusion relation $\subseteq$), then the union of all the linear subspaces belonging to $\mathcal{C}$ is a linear subspace of X having the properties in (5.4). Hence, we can apply Zorn's lemma applies and conclude that there exists at least a maximal element H in $\mathcal{F}$.

2) H is closed in X.
The fact that H and Ω do not intersect gives that H is contained in the

complement of Ω in X. This implies that also its closure $\overline{H}$ does not intersect Ω. Indeed, since Ω is open, we get

$$\overline{H} \subseteq \overline{X \setminus \Omega} = X \setminus \Omega.$$

Then $\overline{H}$ is a linear subspace (as closure of a linear subspace) of X, which is disjoint from Ω and which contains H and so N, i.e. $\overline{H} \in \mathcal{F}$. Hence, as H is maximal in $\mathcal{F}$, it must coincide with its closure. Note that the fact that H is closed guarantees that the quotient space X/H is a Hausdorff t.v.s. (see Proposition 2.3.5).

3) H is an hyperplane

We want to show that H is a hyperplane, i.e. that $dim(X/H) = 1$. To this aim we distinguish the two cases when $\mathbb{K} = \mathbb{R}$ and when $\mathbb{K} = \mathbb{C}$.

3.1) Case $\mathbb{K} = \mathbb{R}$

Let $\phi : X \to X/H$ be the canonical map. Since ϕ is an open linear mapping (see Proposition 2.3.2), $\phi(\Omega)$ is an open convex subset of X/H. Also we have that $\phi(\Omega)$ does not contain the origin $\hat{o}$ of X/H. Indeed, if $\hat{o} \in \phi(\Omega)$ holds, then there would exist $x \in \Omega$ s.t. $\phi(x) = \hat{o}$ and so $x \in H$, which would contradict the assumption $H \cap \Omega = \emptyset$. Let us set:

$$A = \bigcup_{\lambda > 0} \lambda \phi(\Omega).$$

Then the subset A of X/H is open, convex and it is a cone which does not contain the origin $\hat{o}$.

Let us observe that X/H has at least dimension 1. If $dim(X/H) = 0$ then $X/H = \{\hat{o}\}$ and so $X = H$ which contradicts the fact that Ω does not intersect H (recall that we assumed Ω is non-empty). Suppose that $dim(X/H) \geq 2$, then to get our conclusion it will suffice to show the following claims:

Claim 1: The boundary ∂A of A must contain at least one point $x \neq \hat{o}$.

Claim 2: The point $-x$ cannot belong to A.

In fact, once Claim 1 is established, we have that $x \notin A$, because $x \in \partial A$ and A is open. This together with Claim 2 gives that both x and $-x$ belong to the complement of A in X/H and, therefore, so does the straight line L defined by these two points. (If there was a point $y \in L \cap A$ then any positive multiple of y would belong to $L \cap A$, as A is a cone. Hence, for some $\lambda > 0$ we would have $x = \lambda y \in L \cap A$, which contradicts the fact that $x \notin A$.) Then:

- $\phi^{-1}(L)$ is a linear subspace of X
- $\phi^{-1}(L) \cap \Omega = \emptyset$, since $L \cap A = \emptyset$
- $\phi^{-1}(L) \supsetneq H$ because $\hat{o} = \phi(H) \subseteq L$ but $L \neq \{\hat{o}\}$ since $x \neq \hat{o}$ is in L.

This contradicts the maximality of H and so $dim(X/H) = 1$.
To complete the proof of 3.1) let us show the two claims.

Proof. of Claim 1
Suppose that $\partial A = \{\hat{o}\}$. This means that A has empty boundary in the set $(X/H) \setminus \{\hat{o}\}$ and so that A is a connected component of $(X/H) \setminus \{\hat{o}\}$. Since $dim(X/H) \geq 2$, the space $(X/H) \setminus \{\hat{o}\}$ is arc-connected and so it is itself a connected space. Hence, $A = (X/H) \setminus \{\hat{o}\}$ which contradicts the convexity of A since $(X/H) \setminus \{\hat{o}\}$ is clearly not convex. □

Proof. of Claim 2
Suppose $-x \in A$. Then, as A is open, there is a neighborhood V of $-x$ entirely contained in A. This implies that $-V$ is a neighborhood of x. Since x is a boundary point of A, there exists $y \in (-V) \cap A$. But then $-y \in V \subset A$ and so, by the convexity of A, the whole line segment between y and $-y$ is contained in A, in particular $\hat{o}$, which contradicts the definition of A. □

3.2) Case $\mathbb{K} = \mathbb{C}$
Although here we are consider the scalars to be the complex numbers, we may view X as a vector space over the real numbers and it is obvious that its topology, as originally given, is still compatible with its linear structure. By step 3.1) above, we know that there exists a real hyperplane H_0 of X which contains N and does not intersect Ω. By a real hyperplane, we mean that H_0 is a linear subspace of X viewed as a vector space over the field of real numbers such that $dim_{\mathbb{R}}(X/H) = 1$.

Now it is easy to see that $iN = N$ (here $i = \sqrt{-1}$). Hence, setting $H := H_0 \cap iH_0$, we have that $N \subseteq H$ and $H \cap \Omega = \emptyset$. Then to complete the proof it remains to show that this H is a complex hyperplane. It is obviously a complex linear subspace of X and its real codimension is ≥ 1 and ≤ 2 (since the intersection of two distinct hyperplanes is always a linear subspace with codimension two). Hence, its complex codimension is equal to one.

□

Proof. Theorem 5.1.1
Let p be a seminorm on the vector space X, M a linear subspace of X, and f a linear functional defined on M fulfilling (5.1). As already remarked before, this means that f is continuous on M w.r.t. the topology induced by p on X (which makes X a l.c. t.v.s.).

Consider the subset $N := \{x \in M : f(x) = 1\}$. Taking any vector $x_0 \in N$, it is easy to see that $N - x_0 = Ker(f)$ (i.e. the kernel of f in M), which is a hyperplane of M and so a linear subspace of X. Therefore, setting $M_0 := N - x_0$, we have the following decomposition of M:

$$M = M_0 \oplus \mathbb{K}x_0,$$

where $\mathbb{K}x_0$ is the one-dimensional linear subspace through x_0. In other words

$$\forall\, x \in M,\ \exists!\, \lambda \in \mathbb{K}, y \in M_0 :\ x = y + \lambda x_0.$$

Then

$$\forall\, x \in M,\ f(x) = f(y) + \lambda f(x_0) = \lambda f(x_0) = \lambda,$$

which means that the values of f on M are completely determined by the ones on N. Consider now the open unit semiball of p:

$$U := \mathring{U}_p = \{x \in X : p(x) < 1\},$$

which we know being an open convex subset of X endowed with the topology induced by p. Then $N \cap U = \emptyset$ because if there was $x \in N \cap U$ then $p(x) < 1$ and $f(x) = 1$, which contradict (5.1).

By Theorem 5.1.2 (affine version), there exists a closed affine hyperplane H of X with the property that $N \subseteq H$ and $H \cap U = \emptyset$. Then $H - x_0$ is a hyperplane and so the kernel of a continuous linear functional $\tilde{f}$ on X non-identically zero.

Arguing as before (consider here the decomposition $X = (H - x_0) \oplus \mathbb{K}x_0$), we can deduce that the values of $\tilde{f}$ on X are completely determined by the ones on N and so on H (because for any $h \in H$ we have $h - x_0 \in Ker(\tilde{f})$ and so $\tilde{f}(h) - \tilde{f}(x_0) = \tilde{f}(h - x_0) = 0$). Since $\tilde{f} \not\equiv 0$, we have that $\tilde{f}(x_0) \neq 0$ and w.l.o.g. we can assume $\tilde{f}(x_0) = 1$ i.e. $\tilde{f} \equiv 1$ on H. Therefore, for any $x \in M$ there exist unique $\lambda \in \mathbb{K}$ and $y \in N - x_0 \subseteq H - x_0$ s.t. $x = y + \lambda x_0$, we get that:

$$\tilde{f}(x) = \lambda \tilde{f}(x_0) = \lambda = \lambda f(x_0) = f(x),$$

i.e. f is the restriction of $\tilde{f}$ to M. Furthermore, the fact that $H \cap U = \emptyset$ means that $\tilde{f}(x) = 1$ implies $p(x) \geq 1$. Then for any $y \in X$ s.t. $\tilde{f}(y) \neq 0$ we have that: $\tilde{f}\left(\frac{y}{\tilde{f}(y)}\right) = 1$ and so that $p\left(\frac{y}{\tilde{f}(y)}\right) \geq 1$ which implies that $|\tilde{f}(y)| \leq p(y)$. The latter obviously holds for $\tilde{f}(y) = 0$. Hence, (5.2) is established. □

5.2 Applications of Hahn-Banach theorem

The Hahn-Banach theorem is frequently applied in analysis, algebra and geometry, as will be seen in the forthcoming course. We will briefly indicate in this section some applications of this theorem to problems of separation of convex sets and to the multivariate moment problem. From now on we will focus on t.v.s. over the field of real numbers.

5.2.1 Separation of convex subsets of a real t.v.s.

Let X t.v.s. over the field of real numbers and H be a closed affine hyperplane of X. We say that two disjoint subsets A and B of X are *separated* by H if A is contained in one of the two closed half-spaces determined by H and B is contained in the other one. We can express this property in terms of functionals. Indeed, since $H = L^{-1}(\{a\})$ for some $L : X \to \mathbb{R}$ linear not identically zero and some $a \in \mathbb{R}$, we can write that A and B are separated by H if and only if:

$$\exists\, a \in \mathbb{R} \text{ s.t. } L(A) \geq a \text{ and } L(B) \leq a.$$

where for any $S \subseteq X$ the notation $L(S) \leq a$ simply means $\forall s \in S, L(s) \leq a$ (and analogously for $\geq, <, >, =, \neq$).
We say that A and B are *strictly separated* by H if at least one of the two inequalities is strict. (Note that there are several definition in literature for the strict separation but for us it will be just the one defined above) In the present subsection we would like to investigate whether one can separate, or strictly separate, two disjoint convex subsets of a real t.v.s..

Proposition 5.2.1. *Let X be a t.v.s. over the real numbers and A, B two disjoint convex subsets of X.*

a) If A is open nonempty and B is nonempty, then there exists a closed affine hyperplane H of X separating A and B, i.e. there exists $a \in \mathbb{R}$ and a functional $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq a$ and $L(B) \leq a$.

b) If in addition B is open, the hyperplane H can be chosen so as to strictly separate A and B, i.e. there exists $a \in \mathbb{R}$ and $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq a$ and $L(B) < a$.

c) If A is a cone and B is open, then a can be chosen to be zero, i.e. there exists $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq 0$ and $L(B) < 0$.

Proof.

a) Consider the set $A - B := \{a - b : a \in A, b \in B\}$. Then: $A - B$ is an open subset of X as it is the union of the open sets $A - y$ as y varies over B; $A - B$ is convex as it is the Minkowski sum of the convex sets A and $-B$; and $o \notin (A - B)$ because if this was the case then there would be at least a point in the intersection of A and B which contradicts the assumption that they are disjoint. By applying Theorem 5.1.2 to $N = \{o\}$ and $U = A - B$ we have that there is a closed hyperplane H of X which does not intersect $A - B$ (and passes through the origin) or, which is equivalent, there exists a linear form f on X not identically zero such that $f(A-B)\neq 0$. Then there exists a linear form L on X not identically zero such that $L(A-B)>0$ (in the case $f(A-B)<0$ just take $L:=-f$) i.e.

$$\forall\, x\in A, \forall\, y\in B,\;\; L(x)>L(y). \tag{5.5}$$

Since $B\neq\emptyset$ we have that $a:=\inf_{x\in A}L(x)>-\infty$. Then (5.5) implies that $L(B)\leq a$ and we clearly have $L(A)\geq a$.

b) Let now both A and B be open convex and nonempty disjoint subsets of X. By part a) we have that there exists $a\in\mathbb{R}$ and $L:X\to\mathbb{R}$ linear not identically zero s.t. $L(A)\geq a$ and $L(B)\leq a$. Suppose that there exists $b\in B$ s.t. $L(b)=a$. Since B is open, for any $x\in X$ there exists $\varepsilon>0$ s.t. for all $t\in[0,\varepsilon]$ we have $b+tx\in B$. Therefore, as $L(B)\leq a$, we have that

$$L(b+tx)\leq a, \forall\, t\in[0,\varepsilon]. \tag{5.6}$$

Now fix $x\in X$, consider the function $f(t):=L(b+tx)$ for all $t\in\mathbb{R}$ whose first derivative is clearly given by $f'(t)=L(x)$ for all $t\in\mathbb{R}$. Then (5.6) means that $t=0$ is a point of local minimum for f and so $f'(0)=0$ i.e. $L(x)=0$. As x is an arbitrary point of x, we get $L\equiv 0$ on X which is a contradiction. Hence, $L(B)<a$.

c) Let now A be a nonempty convex cone of X and B an open convex nonempty subset of X s.t. $A\cap B=\emptyset$. By part a) we have that there exists $a\in\mathbb{R}$ and $L:X\to\mathbb{R}$ linear not identically zero s.t. $L(A)\geq a$ and $L(B)\leq a$. Since A is a cone, for any $t>0$ we have that $tA\subseteq A$ and so $tL(A)=L(tA)\geq a$ i.e. $L(A)\geq\frac{a}{t}$. This implies that $L(A)\geq\inf_{t>0}\frac{a}{t}=0$. Moreover, part a) also gives that $L(B)<L(A)$. Therefore, for any $t>0$ and any $x\in A$, we have in particular $L(B)<L(tx)=tL(x)$ and so $L(B)\leq\inf_{t>0}tL(x)=0$. Since B is also open, we can exactly proceed as in part b) to get $L(B)<0$.

□

Let us show now two interesting consequences of this result which we will use in the following subsection.

Corollary 5.2.2. *Let X be a vector space over $\mathbb{R}$ endowed with the finest locally convex topology φ. If C is a nonempty closed cone in X and $x_0\in X\setminus C$ then there exists a linear functional $L:X\to\mathbb{R}$ non identically zero s.t. $L(C)\geq 0$ and $L(x_0)<0$.*

Proof. As C is closed in (X,φ) and $x_0\in X\setminus C$, we have that $X\setminus C$ is an open neighbourhood of x_0. Then the local convexity of (X,φ) guarantees that there exists an open convex neighbourhood V of x_0 s.t. $V \subseteq X \setminus C$ i.e. $V \cap C = \emptyset$. By Proposition 5.2.1-c), we have that there exists $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(\mathring{C}) \geq 0$ and $L(V) < 0$, in particular $L(x_0) < 0$. □

Before giving the second corollary, let us introduce some notations. Given a cone C in a t.v.s. (X, τ) we define the first and the second dual of C w.r.t. τ respectively as follows:

$$C_\tau^\vee := \{\ell : X \to \mathbb{R} \text{ linear} \,|\, \ell \text{ is} \, \tau - \text{continuous and } \ell(C) \geq 0\}$$

$$C_\tau^{\vee\vee} := \{x \in X \,|\forall\, \ell \in C_\tau^\vee,\, \ell(x) \geq 0\}.$$

Corollary 5.2.3. *Let X be a vector space over $\mathbb{R}$ endowed with the finest locally convex topology φ. If C is a nonempty cone in X, then $\overline{C}^\varphi = C_\varphi^{\vee\vee}$.*

Proof. Let us first observe that $C \subseteq C_\varphi^{\vee\vee}$, because for any $x \in C$ and any $\ell \in C_\varphi^\vee$ we have by definition of first dual of C that $\ell(x) \geq 0$ and so that $x \in C_\varphi^{\vee\vee}$. Then we get that $\overline{C}^\varphi \subseteq \overline{C_\varphi^{\vee\vee}}^\varphi$. But $C_\varphi^{\vee\vee}$ is closed since $C_\varphi^{\vee\vee} = \bigcap_{\ell \in C_\varphi^\vee} \ell([0, +\infty))$ and each $\ell \in C_\varphi^\vee$ is φ-continuous. Hence, $\overline{C}^\varphi \subseteq C_\varphi^{\vee\vee}$.

Conversely, suppose there exists $x_0 \in C_\varphi^{\vee\vee} \setminus \overline{C}^\varphi$. By Corollary 5.2.2, there exists a linear functional $L : X \to \mathbb{R}$ non identically zero s.t. $L(\overline{C}^\varphi) \geq 0$ and $L(x_0) < 0$. As $L(C) \geq 0$ and every linear functional is $\varphi-$continuous, we have $L \in C_\varphi^\vee$. This together with the fact that $L(x_0) < 0$ give $x_0 \notin C_\varphi^{\vee\vee}$, which is a contradiction. Hence, $\overline{C}^\varphi = C_\varphi^{\vee\vee}$. □

5.2.2 Multivariate real moment problem

Let $d \in \mathbb{N}$ and let $\mathbb{R}[\underline{x}]$ be the ring of polynomials with real coefficients and d variables $\underline{x} := (x_1, \ldots, x_d)$. Fixed a subset K of $\mathbb{R}^d$, we denote by

$$Psd(K) := \{p \in \mathbb{R}[\underline{x}] : p(\underline{x}) \geq 0, \forall x \in K\}.$$

Definition 5.2.4 (Multivariate real $K-$moment problem)**.**
Given a closed subset K of $\mathbb{R}^d$ and a linear functional $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$, does there exists a nonnegative finite Borel measure μ s.t.

$$L(p) = \int_{\mathbb{R}^d} p(\underline{x})\mu(d\underline{x}), \,\forall p \in \mathbb{R}[\underline{x}]$$

and $supp(\mu) \subseteq K$ (where $supp(\mu)$ denotes the support of the measure μ)?

*If such a measure exists, we say that μ is a K-*representing measure *for L and that it is a solution to the $K-$moment problem for L.*

A necessary condition for the existence of a solution to the $K-$moment problem for the linear functional L is clearly that L is nonnegative on $Psd(K)$. In fact, if there exists a representing measure μ for L then for all $p \in Psd(K)$ we have

$$L(p) = \int_{\mathbb{R}^d} p((\underline{x}))\mu(d\underline{x}) = \int_K p((\underline{x}))\mu(d\underline{x}) \geq 0$$

since μ is nonnegative and supported on K and p is nonnegative on K.

It is then natural to ask if the nonnegative of L on $Psd(K)$ is also sufficient. The answer is positive and it was established by Riesz in 1923 for $d = 1$ and by Haviland for any $d \geq 2$.

Theorem 5.2.5 (Riesz-Haviland Theorem). *Let K be a closed subset of $\mathbb{R}^d$ and $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ be linear. L has a $K-$representing measure if and only if $L(Psd(K)) \geq 0$.*

Note that this theorem provides a complete solution for the $K-$ moment problem but it is quite unpractical! In fact, it reduces the solvability of the $K-$moment problem to the problem of classifying all polynomials which are nonnegative on a prescribed closed subset K of $\mathbb{R}^d$ i.e. to characterize $Psd(K)$. This is actually a hard problem to be solved for general K and it is a core question in real algebraic geometry. For example, if we think of the case $K = \mathbb{R}^d$ then for $d = 1$ we know that $Psd(K) = \sum \mathbb{R}[\underline{x}]^2$, where $\sum \mathbb{R}[\underline{x}]^2$ denotes the set of squares of polynomials. However, for $d \geq 2$ this equality does not hold anymore as it was proved by Hilbert in 1888. It is now clear that to make the conditions of the Riesz-Haviland theorem actually checkable we need to be able to write/approximate a non-negative polynomial on K in a way that makes its non-negativity apparent, i.e. as a sum of squares or as an element of quadratic modules of $\mathbb{R}[\underline{x}]$. For a special class of closed subsets of $\mathbb{R}^d$ we actually have such representations and we can get better conditions than the one of Riesz-Haviland type to solve the $K-$moment problem.

Definition 5.2.6. *Given a finite set of polynomials $S := \{g_1, \ldots, g_s\}$, we call the* basic closed semialgebraic set *generated by S the following*

$$K_S := \{\underline{x} \in \mathbb{R}^d : g_i(\underline{x}) \geq 0, \, i = 1, \ldots, s\}.$$

Definition 5.2.7. *A subset M of $\mathbb{R}[\underline{x}]$ is said to be a* quadratic module *if $1 \in M$, $M + M \subseteq M$ and $h^2 M \subseteq M$ for any $h \in \mathbb{R}[\underline{x}]$.*

Note that each quadratic module is a cone in $\mathbb{R}[\underline{x}]$.

Definition 5.2.8. *A quadratic module M of $\mathbb{R}[\underline{x}]$ is called* Archimedean *if there exists $N \in \mathbb{N}$ s.t. $N - (\sum_{i=1}^d x_i^2) \in M$.*

For $S := \{g_1, \ldots, g_s\}$ finite subset of $\mathbb{R}[\underline{x}]$, we define the *quadratic module generated by S* to be:

$$M_S := \left\{ \sum_{i=0}^{s} \sigma_i g_i : \sigma_i \in \sum \mathbb{R}[\underline{x}]^2, i = 0, 1, \ldots, s \right\},$$

where $g_0 := 1$.

Remark 5.2.9. *Note that $M_S \subseteq Psd(K_S)$ and M_S is the smallest quadratic module of $\mathbb{R}[\underline{x}]$ containing S.*

Consider now the finite topology on $\mathbb{R}[\underline{x}]$ (see Definition 4.5.1) which we have proved to be the finest locally convex topology on this space (see Proposition 4.5.2) and which we therefore denote by φ. By Corollary 5.2.3, we get that

$$\overline{M_S}^{\varphi} = (M_S)_{\varphi}^{\vee\vee} \tag{5.7}$$

Moreover, the *Putinar Positivstellesatz* (1993), a milestone result in real algebraic geometry, provides that if M_S is Archimedean then

$$Psd(K_S) \subseteq \overline{M_S}^{\varphi}. \tag{5.8}$$

Note that M_S is Archimedean implies that K_S is compact while the converse is in general not true (see e.g. M. Marshall, Positive polynomials and sum of squares, 2008).

Combining (5.7) and (5.8), we get the following result.

Proposition 5.2.10. *Let $S := \{g_1, \ldots, g_s\}$ be a finite subset of $\mathbb{R}[\underline{x}]$ and $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ linear. Assume that M_S is Archimedean. Then there exists a K_S-representing measure μ for L if and only if $L(M_S) \geq 0$, i.e. $L(h^2 g_i) \geq 0$ for all $h \in \mathbb{R}[\underline{x}]$ and for all $i \in \{1, \ldots, s\}$.*

Proof. Suppose that $L(M_S) \geq 0$ and let us consider the finite topology φ on $\mathbb{R}[\underline{x}]$. Then the linear functional L is φ-continuous and so $L \in (M_S)_{\varphi}^{\vee}$. Moreover, as M_S is assumed to be Archimedean we have

$$Psd(K_S) \overset{(5.8)}{\subseteq} \overline{M_S}^{\varphi} \overset{(5.7)}{=} (M_S)_{\varphi}^{\vee\vee}.$$

Since any $p \in Psd(K_S)$ is also an element of $(M_S)_{\varphi}^{\vee\vee}$, we have that for any $\ell \in (M_S)_{\varphi}^{\vee}$, $\ell(Psd(K_S)) \geq 0$ and in particular $L(Psd(K_S)) \geq 0$. Hence, by Riesz-Haviland theorem we get the existence of a K_S-representing measure μ for L.

Conversely, suppose that the there exists a K_S-representing measure μ for L. Then for all $p \in M_S$ we have in particular that

$$L(p) = \int_{\mathbb{R}^d} p(\underline{x})\mu(d\underline{x})$$

which is nonnegative as μ is a nonnegative measure supported on K_S and $p \in M_S \subseteq Psd(K_S)$. □

From this result and its proof we understand that whenever we know that $Psd(K_S) \subseteq \overline{M_S}^{\varphi}$, we need to check only that $L(M_S) \geq 0$ to find out whether there exists a solution for the K_S-moment problem for L. Then it makes sense to look for closure results of this kind in the case when M_S is not Archimedean and so we cannot apply the Putinar Positivstellesatz. Actually whenever we know that $Psd(K_S) \subseteq \overline{M_S}^{\tau}$ where τ is a locally convex topology on $\mathbb{R}[\underline{x}]$, the condition $L(M_S) \geq 0$ is necessary and sufficient for the existence of a solution of the K_S-moment problem for any $\tau-$continuous functional on $\mathbb{R}[\underline{x}]$ (see M. Ghasemi, S. Kuhlmann, E. Samei, 2012). This relationship between the closure of quadratic modules and the representability of functionals continuous w.r.t. locally convex topologies started a new research line in the study of the moment problem which is still bringing interesting results.