Chapter 3

$K-$Moment Problem: the operator theoretical approach

3.1 Basics from spectral theory of bounded operators

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space (i.e. a complete inner product space). We denote by $\| \cdot \|$ the norm induced on $\mathcal{H}$ by the inner product $\langle \cdot, \cdot \rangle$.

Definition 3.1.1. *An* operator *T on $\mathcal{H}$ is a linear map from a linear subspace $\mathcal{D}(T)$ of $\mathcal{H}$ (called the* domain *of T) into $\mathcal{H}$. We say that*

- *T is* bounded *if its operator norm $\|T\|_{op} := \sup_{x \in \mathcal{D}(T) \setminus \{o\}} \frac{\|Tx\|}{\|x\|}$ is finite.*
- *T is* symmetric *if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in \mathcal{D}(T)$.*

Definition 3.1.2. *Let T be a bounded operator with $\mathcal{D}(T) = \mathcal{H}$. Then*

- *the unique bounded operator $T^* : \mathcal{H} \to \mathcal{H}$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in \mathcal{H}$ is called the* adjoint *of T.*
- *T is called* self-adjoint *if $T = T^*$.*

Note that a bounded operator defined on the whole $\mathcal{H}$ is self-adjoint if and only if it is symmetric.

Definition 3.1.3. *Two operators T_1, T_2 defined on the same Hilbert space $\mathcal{H}$* commute *if $T_1 T_2 x = T_2 T_1 x$ for all $x \in \mathcal{H}$.*

Theorem 3.1.4 (Spectral Theorem for bounded operators)**.** *Let $T_1, \ldots, T_n$ be n pairwise commuting bounded self-adjoint operators having as domain the same separable Hilbert space $\mathcal{H}$ and let $v \in \mathcal{H}$. Then there exists a unique non-negative Radon measure μ_v on $\mathbb{R}^n$ such that*

$$\langle v, T_1^{\alpha_1} \cdots T_n^{\alpha_n} v \rangle = \int_{\mathbb{R}^n} \underline{X}^\alpha d\mu_v < \infty, \ \forall \ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$$

and μ_v is supported in $B_{\|T_1\|_{op}}(0) \times \cdots \times B_{\|T_n\|_{op}}(0)$ where $B_R(0)$ denotes the closed ball of radius R and center 0 in $\mathbb{R}$.

(for a proof see e.g. [40, Theorem 5.23]).

Let us also recall a fundamental theorem about linear transformations on normed spaces (see e.g. [36, Theorem I.7]), which will be useful in the following.

Theorem 3.1.5 (Bounded Linear Transformation Theorem)**.** *Let Y be a Banach space, Z be a normed space, and U a dense subset of Z. If $\varphi : U \to Y$ is a bounded linear map, then φ can be uniquely extended to a bounded linear map $\overline{\varphi} : Z \to Y$ and $\|\overline{\varphi}\|_{op} = \|\varphi\|_{op}$*

3.2 Solving the KMP for K compact semialgebraic sets

In Section 2.3 we proved the celebrated solution to the KMP for K compact due to Schmüdgen, see Corollary 2.3.17, by combining Schmüdgen Nichtnegativstellensatz and Riesz'-Haviland Theorem. In this section we are going to provide the original proof given by Schmüdgen in [39], which is based on an operator theoretical approach to the moment problem.

Theorem 3.2.1. *Let $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ linear and $S := \{g_1, \dots, g_s\} \subset \mathbb{R}[\underline{X}]$ such that the associated bcsas K_S is compact. Then there exists a K_S-representing measure for L if and only if $L(h^2 g_1^{e_1} \cdots g_s^{e_s}) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$, $e_1, \dots, e_s \in \{0, 1\}$.*

Proof. Suppose there exists a K_S-representing measure μ for L, then for any $h \in \mathbb{R}[\underline{X}]$ and any $e_1, \dots, e_s \in \{0, 1\}$ we have

$$L(h^2 g_1^{e_1} \cdots g_s^{e_s}) = \int_{K_S} h^2 g_1^{e_1} \cdots g_s^{e_s} d\mu,$$

which is non-negative as integral of a non-negative function w.r.t. a non-negative measure.

Conversely, suppose that $L(h^2 g_1^{e_1} \cdots g_s^{e_s}) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$, $e_1, \dots, e_s \in \{0, 1\}$, i.e. $L(T_S) \subseteq [0, +\infty)$ where T_S is the preordering generated by S. We want to show the existence of a K_S-representing measure by using the Spectral Theorem 3.1.4.

First of all, let us observe that the compactness of K_S implies that there exists $\sigma > 0$ such that for any $x \in K_S$ we have $|x|^2 := x_1^2 + \cdots + x_n^2 < \sigma^2$, i.e. $\sigma^2 - |x|^2 > 0, \forall\, x \in K_S$. Hence, by Stengle Striktpositivstellensatz 1.3.1, we have that

$$\exists\, p, q \in T_S \text{ s.t. } (\sigma^2 - |x|^2)p = 1 + q. \tag{3.1}$$

Consider now the symmetric bilinear form

$$\begin{array}{rccl} \langle\ ,\ \rangle: & \mathbb{R}[\underline{X}] \times \mathbb{R}[\underline{X}] & \to & \mathbb{R} \\ & (\ p\ ,\ q\ ) & \mapsto & \langle p, q\rangle := L(pq) \end{array}$$

(note that $\langle \cdot, \cdot \rangle$ coincides with $\langle \cdot, \cdot \rangle_1$ as in Definition 2.3.9).

This is a quasi-inner product, since for any $f \in \mathbb{R}[\underline{X}]$ we have by assumption that $\langle f, f\rangle = L(f^2) \geq 0$ but $\langle f, f\rangle = 0$ does not necessarily imply that $f \equiv 0$ (e.g. if $L : \mathbb{R}[X] \to \mathbb{R}$ is linear s.t. $L(\underline{X}^n) = 1$ for $n = 0$ and $L(\underline{X}^n) = 0$ for $n \in \mathbb{N}$, then $\langle \underline{X}, \underline{X}\rangle = L(\underline{X}^2) = 0$ but $\underline{X}$ is not the zero polynomial.)

Let us consider the ideal $N := \{f \in \mathbb{R}[\underline{X}] : L(f^2) = 0\}$. Hence, there exists a well-defined inner product on the quotient vector space $\mathbb{R}[\underline{X}]/N$ which, by abuse of notation, we denote again by $\langle \cdot, \cdot \rangle$ and that is defined by

$$\langle f + N, r + N\rangle := L(fr), \forall f, r \in \mathbb{R}[\underline{X}]. \tag{3.2}$$

Let us denote by $\mathcal{H}_L$ the Hilbert space obtained by taking the completion of $\mathbb{R}[\underline{X}]/N$ w.r.t. the inner product $\langle \cdot, \cdot \rangle$ in (3.2) and by $\| \cdot \|$ the norm on $\mathcal{H}_L$ induced by $\langle \cdot, \cdot \rangle$.

<u>*Claim*</u>: $\forall\ h \in \mathbb{R}[\underline{X}],\ j \in \{1, \ldots, n\},\ \ \|X_j h + N\| \leq \sigma \|h + N\|.$

Proof. of Claim Let us fix $h \in \mathbb{R}[\underline{X}]$ and $d \in \mathbb{N}$. Take p and q as in (3.1) and defined $|\underline{X}|^2 := X_1^2 + \cdots + X_n^2$. Since $(1 + q)|\underline{X}|^{2d-2}h^2 \in T_S$ and L is non-negative on elements of T_S, we have that:

$$\begin{aligned} L(|\underline{X}|^{2d}h^2 p) &\leq L(|\underline{X}|^{2d}h^2 p) + L\left((1 + q)|\underline{X}|^{2d-2}h^2\right) \\ &= L\left(|\underline{X}|^{2d-2}h^2(|\underline{X}|^2 p + 1 + q)\right) \\ &\overset{(3.1)}{=} L\left(|\underline{X}|^{2d-2}h^2\sigma^2 p\right) \\ &= \sigma^2 L\left(|\underline{X}|^{2(d-1)}h^2 p\right). \end{aligned}$$

Iterating, we get that

$$\forall\ d \in \mathbb{N},\ L(|\underline{X}|^{2d}h^2 p) \leq \sigma^{2d} L(h^2 p). \tag{3.3}$$

Fix $j \in \{1, \ldots, n\}$ and consider $\ell_j : \mathbb{R}[X_j] \to \mathbb{R}$ defined by $\ell_j(r) := L(rh^2)$, for all $r \in \mathbb{R}[X_j]$. Then ℓ_j is linear and $\ell_j(r^2) = L(r^2h^2) = L((rh)^2) \geq 0$, since by assumption L is non-negative on squares. Then, by Hamburger's

Theorem 2.3.2 we have that there exists an $\mathbb{R}-$representing measure $\nu_{h,j}$ for ℓ_j. Therefore, for any $\lambda > 0$ and any $d \in \mathbb{N}$ we have

$$\begin{aligned}
\int_{(-\infty,-\lambda)\cup(\lambda,+\infty)} \lambda^{2d} d\nu_{h,j} &\leq \int_{(-\infty,-\lambda)\cup(\lambda,+\infty)} X_j^{2d} d\nu_{h,j} \\
&\leq \int_{\mathbb{R}} X_j^{2d} d\nu_{h,j} = \ell_j(X_j^{2d}) = L(X_j^{2d}h^2) \\
&\leq L\left(X_j^{2d}h^2(|\underline{X}|^2 p + 1 + q)\right) \\
&\overset{(3.1)}{=} L(X_j^{2d}h^2\sigma^2 p) = \sigma^2 L(X_j^{2d}h^2 p) \\
&\leq \sigma^2 L(|\underline{X}|^{2d}h^2 p) \overset{(3.3)}{\leq} \sigma^{2+2d} L(h^2 p).
\end{aligned}$$

Hence, we proved that for any $\lambda > 0$ and any $d \in \mathbb{N}$ we have

$$\int_{(-\infty,-\lambda)\cup(\lambda,+\infty)} d\nu_{h,j} \leq \left(\frac{\sigma}{\lambda}\right)^{2d} \sigma^2 L(h^2 p).$$

In particular, if we take $\lambda > \sigma$ and $d \to \infty$, then $\int_{(-\infty,-\lambda)\cup(\lambda,+\infty)} d\nu_{h,j} = 0$ and so that $\nu_{h,j}$ is supported in $[-\sigma, \sigma]$. Then

$$\begin{aligned}
\|X_j h + N\|^2 &= L(X_j^2 h^2) = \ell_j(X_j^2) = \int_{\mathbb{R}} X_j^2 d\nu_{h,j} = \int_{[-\sigma,\sigma]} X_j^2 d\nu_{h,j} \\
&\leq \sigma^2 \int_{[-\sigma,\sigma]} d\nu_{h,j} = \sigma^2 \ell_j(1) = \sigma^2 L(h^2) = \sigma^2 \|h + N\|^2.
\end{aligned}$$

□(Claim)

For any $j \in \{1, \ldots, n\}$, let us define the *multiplication operator* as follows

$$\begin{aligned}
W_j : \quad \mathbb{R}[\underline{X}]/N &\to \mathbb{R}[\underline{X}]/N \\
h + N &\mapsto X_j h + N
\end{aligned}$$

This is a well-defined operator with s.t. $\mathcal{D}(W_j) = \mathbb{R}[\underline{X}]/N$ is dense in $\mathcal{H}_L$ and
(a) $\underline{W_j \text{ is bounded}}$, since

$$\|W_j\|_{op} := \sup_{\substack{r \in \mathcal{D}(W_j) \\ r \neq o}} \frac{\|W_j r\|}{\|r\|} = \sup_{\substack{h \in \mathbb{R}[\underline{X}] \\ h \notin N}} \frac{\|X_j h + N\|}{\|h + N\|} \overset{\text{Claim}}{\leq} \sigma \sup_{\substack{h \in \mathbb{R}[\underline{X}] \\ h \notin N}} \frac{\|h + N\|}{\|h + N\|} = \sigma.$$

As $(\mathbb{R}[\underline{X}]/N, \|\cdot\|)$ is a normed space, this means that W_j is continuous.

(b) W_j is symmetric, since for any $h, r \in \mathbb{R}[\underline{X}]/N$ we have

$$\langle W_j h, r\rangle = L(X_j hr) = L(hX_j r) = \langle h, W_j r\rangle.$$

(c) $W_1, \ldots, W_n$ are pairwise commuting, since for any $j \neq k$ in $\{1, \ldots, n\}$ and any $h \in \mathbb{R}[\underline{X}]$ we have

$$W_j W_k(h+N) = W_j(X_k h+N) = X_j X_k h+N = X_k X_j h+N = W_k W_j(h+N).$$

By Theorem 3.1.5 (applied for $Z = Y = \mathcal{H}_L$, $U = \mathbb{R}[\underline{X}]/N$, $\varphi = W_j$), there exists a unique bounded operator $\overline{W_j} : \mathcal{H}_L \to \mathcal{H}_L$ extending W_j and $\|\overline{W_j}\|_{op} = \|W_j\|_{op}$. Since each $\mathcal{D}(W_j)$ is dense in $\mathcal{H}_L$ and each W_j is bounded (so continuous), we have that properties (b) and (c) above hold also for $\overline{W_1}, \ldots, \overline{W_n}$. Hence, $\overline{W_1}, \ldots, \overline{W_n}$ are pairwise commuting bounded self-adjoint operators with $\mathcal{D}(\overline{W_j}) = \mathcal{H}_L$ for all $j \in \{1, \ldots, n\}$. Then, by the Spectral Theorem 3.1.4, there exists a unique non-negative Radon measure μ such that

$$\langle (1+N), \overline{W_1}^{\alpha_1} \cdots \overline{W_n}^{\alpha_n}(1+N)\rangle = \int_{\mathbb{R}^n} \underline{X}^{\alpha} d\mu < \infty, \forall \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \tag{3.4}$$

and μ is supported in $B_{\|\overline{W_1}\|_{op}}(0) \times \cdots \times B_{\|\overline{W_n}\|_{op}}(0) \overset{(a)}{\subseteq} [-\sigma, \sigma]^n =: Q$.

Since

$$\begin{aligned} \langle (1+N), \overline{W_1}^{\alpha_1} \cdots \overline{W_n}^{\alpha_n}(1+N)\rangle &= \langle (1+N), {W_1}^{\alpha_1} \cdots {W_n}^{\alpha_n}(1+N)\rangle \\ &= \langle (1+N), {X_1}^{\alpha_1} \cdots {X_n}^{\alpha_n} + N\rangle \\ &= L({X_1}^{\alpha_1} \cdots {X_n}^{\alpha_n}) = L(\underline{X}^{\alpha}), \end{aligned}$$

(3.4) becomes

$$L(\underline{X}^{\alpha}) = \int_{\mathbb{R}^n} \underline{X}^{\alpha} d\mu, \forall \alpha \in \mathbb{N}_0^n.$$

Hence, the spectral measure μ is a $Q-$representing measure for L. It remains to show that μ is actually supported on K_S.

For each $i \in \{1, \ldots, n\}$ we have

$$0 \leq L(g_i h^2) = \int_Q g_i h^2 d\mu, \ \forall\ h \in \mathbb{R}[\underline{X}].$$

As Q is compact, we can apply the Stone-Weierstrass Theorem 2.3.27, we get

$$0 \leq L(g_i f^2) = \int_Q g_i f^2 d\mu, \ \forall\ f \in \mathcal{C}(Q).$$

Then

$$0 \leq L(g_i f) = \int_Q g_i f d\mu, \ \forall \ f \in \mathcal{C}(Q) \text{ s.t. } f \geq 0 \text{ on } Q$$

and so the linear functional

$$\begin{array}{rccc} \tilde{L}: & \mathcal{C}(Q) & \to & \mathbb{R} \\ & f & \mapsto & L(g_i f) \end{array}$$

is such that $\tilde{L}(f) \geq 0$ for all $f \geq 0$ on Q. Hence, by Riesz-Markov-Kakutani Theorem 2.2.5, there exists a unique non-negative Radon measure ν such that $\tilde{L}(f) = \int f d\nu$ for all $f \in \mathcal{C}(Q)$. But $\tilde{L}(f) = \int f g_i d\mu$ for all $f \in \mathcal{C}(Q)$, so the signed measure $g_i\mu$ must coincide with ν. Hence, $g_i\mu$ is a non-negative measure, which implies that the support of μ must be contained in the set of non-negativity of each g_i, i.e. μ is supported in K_S.

□