Chapter 3

$K-$Moment Problem: the operator theoretical approach

3.1 Basics from spectral theory

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space (i.e. a complete inner product space). We denote by $\|\cdot\|$ the norm induced on $\mathcal{H}$ by the inner product $\langle\cdot,\cdot\rangle$.

Definition 3.1.1. *An* operator T *on* $\mathcal{H}$ *is a linear map from a linear subspace* $\mathcal{D}(T)$ *of* $\mathcal{H}$ *(called the* domain *of* T*) into* $\mathcal{H}$*. We say that*

- T *is* bounded *if its operator norm* $\|T\|_{op} := \sup_{x\in\mathcal{D}(T)\setminus\{o\}} \frac{\|Tx\|}{\|x\|}$ *is finite.*
- T *is* symmetric *if* $\langle Tx, y\rangle = \langle x, Ty\rangle$ *for all* $x, y \in \mathcal{D}(T)$*.*

3.1.1 Bounded operators

In this subsection we are going to focus on bounded operators defined everywhere in $\mathcal{H}$.

Definition 3.1.2. *Let* T *be a bounded operator with* $\mathcal{D}(T) = \mathcal{H}$*. Then*

- *the unique bounded operator* $T^* : \mathcal{H} \to \mathcal{H}$ *such that* $\langle Tx, y\rangle = \langle x, T^*y\rangle$ *for all* $x, y \in \mathcal{H}$ *is called the* adjoint *of* T*.*
- T *is called* self-adjoint *if* $T = T^*$*.*

Note that a bounded operator defined everywhere in $\mathcal{H}$ is self-adjoint if and only if it is symmetric.

Definition 3.1.3. *Two operators* T_1, T_2 *defined on the same Hilbert space* $\mathcal{H}$ commute *if* $T_1T_2x = T_2T_1x$ *for all* $x \in \mathcal{H}$*.*

Theorem 3.1.4 (Spectral Theorem for bounded operators)**.** *Let* $T_1, \ldots, T_n$ *be* n *pairwise commuting bounded self-adjoint operators having as domain the* *same separable Hilbert space $\mathcal{H}$ and let $v \in \mathcal{H}$. Then there exists a unique non-negative Radon measure μ_v on $\mathbb{R}^n$ such that*

$$\langle v, T_1^{\alpha_1} \cdots T_n^{\alpha_n} v \rangle = \int_{\mathbb{R}^n} \underline{X}^\alpha d\mu_v < \infty, \; \forall \; \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$$

and μ_v is supported in $B_{\|T_1\|_{op}}(0) \times \cdots \times B_{\|T_n\|_{op}}(0)$ where $B_R(0)$ denotes the closed ball of radius R and center 0 in $\mathbb{R}$.

(for a proof see e.g. [38, Chapter VII] and [43, Theorem 5.23]).

Let us also recall a fundamental theorem about linear transformations on normed spaces (see e.g. [38, Theorem I.7]), which will be useful in the following.

Theorem 3.1.5 (Bounded Linear Transformation Theorem)**.** *Let Y be a Banach space, Z be a normed space, and U a dense subset of Z. If $\varphi : U \to Y$ is a bounded linear map, then φ can be uniquely extended to a bounded linear map $\overline{\varphi} : Z \to Y$ and $\|\overline{\varphi}\|_{op} = \|\varphi\|_{op}$*

3.1.2 Unbounded operators

By the Hellinger-Toeplitz theorem, a symmetric operator T with $\mathcal{D}(T) = \mathcal{H}$ is always bounded (see e.g. [38, Section III.5]). Hence, unbounded symmetric operators cannot be defined everywhere in $\mathcal{H}$. For this reason, we need a more general definition of adjoint than the one given for bounded operators.

Definition 3.1.6. *Let $T : \mathcal{D}(T) \to \mathcal{H}$ be linear with $\mathcal{D}(T)$ dense*[1] *in $\mathcal{H}$. Then*

- *the* adjoint *of T is the linear operator T^* with domain*

$$\mathcal{D}(T^*) := \{w \in \mathcal{H} : \exists z_w \in \mathcal{H} \text{ s.t. } \langle Tv, w \rangle = \langle v, z_w \rangle, \quad \forall v \in \mathcal{D}(T)\}$$

  defined by $T^*v = z_w$ for all $v \in \mathcal{D}(T^*)$.
- *T is called* self-adjoint *if $T = T^*$.*

Definition 3.1.7.
Let T_1 and T_2 be two self-adjoint operators with domain in the same Hilbert space $\mathcal{H}$. We say that T_1 and T_2 are strongly commuting *if $e^{ir_1T_1}e^{ir_2T_2} = e^{ir_2T_2}e^{ir_1T_1}$ for all $r_1, r_2 \in \mathbb{R}$.*

Theorem 3.1.8 (Spectral Theorem for unbounded operators)**.**
Let $(T_1, \ldots, T_n)$ be a tuple of self-adjoint operators with domain dense in the same separable Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ which are pairwise strongly commuting

[1] The density of $\mathcal{D}(T)$ in $\mathcal{H}$ ensures that z_w is uniquely determined by the equation $\langle Tv, w \rangle = \langle v, z_w \rangle, \quad \forall v \in \mathcal{D}(T)$.

and let $v \in \mathcal{H}$ be such that $\forall d \in \mathbb{N}_0$, $\forall\, i_1, \ldots, i_{d+1} \in \{1, \ldots, n\}$ we have $T_{i_d} \cdot T_{i_{d-1}} \cdots T_{i_1} v \in \mathcal{D}(T_{i_{d+1}})$ (for $d = 0$ we set T_{i_0} to be the identity operator). Then there exists a unique non-negative Radon measure μ_v such that

$$\langle v, T_{i_d} \cdot T_{i_{d-1}} \cdots T_{i_1} v \rangle = \int_{\mathbb{R}^n} X_{i_1} \cdots X_{i_d} d\mu_v, \ \forall\ d \in \mathbb{N}_0, i_1, \ldots, i_d \in \{1, \ldots, n\}$$

(for a proof see e.g. [38, Section VIII.3] and [43, Theorem 5.23]).

Let us also recall a fundamental result due to Nussbaum dealing with strongly commuting self-adjoint extensions of unbounded symmetric operators. For this we need to defined the notion of quasi-analytic vector for a given linear operator.

Definition 3.1.9.
Let T be a linear operator with $\mathcal{D}(T) \subset \mathcal{H}$. A vector $v \in \mathcal{D}^\infty(T) := \bigcap_{k=1}^{\infty} \mathcal{D}(T^k)$. is said to be quasi-analytic *for T if*

$$\sum_{k=1}^{\infty} |||T^k v|||^{-\frac{1}{k}} = \infty.$$

Theorem 3.1.10.
Let T_1 and T_2 be two unbounded symmetric operators with $\mathcal{D}(T_1)$ and $\mathcal{D}(T_2)$ subsets of the same Hilbert space $\mathcal{H}$. Let $\mathcal{D}$ be a set of vectors in $\mathcal{H}$ which are quasi-analytic for both T_1 and T_2 and such that $T_1\mathcal{D} \subset \mathcal{D}$, $T_2\mathcal{D} \subset \mathcal{D}$, $T_1T_2x = T_2T_1x$ for all $x \in \mathcal{D}$. If the set $\mathcal{D}$ is total in $\mathcal{H}$, i.e. $\overline{\operatorname{span}(\mathcal{D})} = \mathcal{H}$, then there exist unique self-adjoint extensions $\overline{T_1}$ and $\overline{T_2}$ of T_1 and T_2 in $\mathcal{H}$ such that $\overline{T_1}$ and $\overline{T_2}$ are strongly commuting.

(for a proof see e.g. [35, Theorem 6] and [43, Theorem 7.18]).

3.2 Solving the KMP for K compact semialgebraic sets

In Section 2.3 we proved the celebrated solution to the KMP for K compact due to Schmüdgen, see Corollary 2.3.17, by combining Schmüdgen Nichtnegativstellensatz and Riesz'-Haviland Theorem. In this section we are going to provide the original proof given by Schmüdgen in [42], which is based on an operator theoretical approach to the moment problem.

Theorem 3.2.1. *Let $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ linear and $S := \{g_1, \ldots, g_s\} \subset \mathbb{R}[\underline{X}]$ such that the associated bcsas K_S is compact. Then there exists a unique K_S-representing measure for L if and only if $L(h^2 g_1^{e_1} \cdots g_s^{e_s}) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$, $e_1, \ldots, e_s \in \{0, 1\}$.*

Proof.
Suppose there exists a K_S-representing measure μ for L, then for any $h \in \mathbb{R}[\underline{X}]$ and any $e_1, \ldots, e_s \in \{0,1\}$ we have

$$L(h^2 g_1^{e_1} \cdots g_s^{e_s}) = \int_{K_S} h^2 g_1^{e_1} \cdots g_s^{e_s} d\mu,$$

which is non-negative as integral of a non-negative function w.r.t. a non-negative measure.

Conversely, suppose that $L(h^2 g_1^{e_1} \cdots g_s^{e_s}) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$, $e_1, \ldots, e_s \in \{0,1\}$, i.e. $L(T_S) \subseteq [0,+\infty)$ where T_S is the preordering generated by S. We want to show the existence of a K_S-representing measure by using the Spectral Theorem 3.1.4.

First of all, let us observe that the compactness of K_S implies that there exists $\sigma > 0$ such that for any $x \in K_S$ we have $|x|^2 := x_1^2 + \cdots + x_n^2 < \sigma^2$, i.e. $\sigma^2 - |x|^2 > 0, \forall\, x \in K_S$. Hence, by Stengle Striktpositivstellensatz 1.3.1, we have that

$$\exists\; p, q \in T_S \text{ s.t. } (\sigma^2 - |x|^2)p = 1 + q. \tag{3.1}$$

Consider now the symmetric bilinear form

$$\begin{array}{rccc} \langle\ ,\ \rangle : & \mathbb{R}[\underline{X}] \times \mathbb{R}[\underline{X}] & \to & \mathbb{R} \\ & (\ p\ ,\ q\ ) & \mapsto & \langle p, q\rangle := L(pq) \end{array}$$

(note that $\langle \cdot, \cdot \rangle$ coincides with $\langle \cdot, \cdot \rangle_1$ as in Definition 2.3.9).

This is a quasi-inner product, since for any $f \in \mathbb{R}[\underline{X}]$ we have by assumption that $\langle f, f\rangle = L(f^2) \geq 0$ but $\langle f, f \rangle = 0$ does not necessarily imply that $f \equiv 0$ (e.g. if $L : \mathbb{R}[X] \to \mathbb{R}$ is linear s.t. $L(\underline{X}^n) = 1$ for $n = 0$ and $L(\underline{X}^n) = 0$ for $n \in \mathbb{N}$, then $\langle \underline{X}, \underline{X} \rangle = L(\underline{X}^2) = 0$ but $\underline{X}$ is not the zero polynomial.)

Let us consider the ideal $N := \{f \in \mathbb{R}[\underline{X}] : L(f^2) = 0\}$. Hence, there exists a well-defined inner product on the quotient vector space $\mathbb{R}[\underline{X}]/N$ which, by abuse of notation, we denote again by $\langle \cdot, \cdot \rangle$ and that is defined by

$$\langle f + N, r + N \rangle := L(fr), \forall f, r \in \mathbb{R}[\underline{X}]. \tag{3.2}$$

Let us denote by $\mathcal{H}_L$ the Hilbert space obtained by taking the completion of $\mathbb{R}[\underline{X}]/N$ w.r.t. the inner product $\langle \cdot, \cdot \rangle$ in (3.2)[2] and by $\| \cdot \|$ the norm on $\mathcal{H}_L$ induced by $\langle \cdot, \cdot \rangle$.

Claim: $\forall\; h \in \mathbb{R}[\underline{X}],\; j \in \{1, \ldots, n\},\quad \|X_j h + N\| \leq \sigma \|h + N\|.$

[2] This construction is actually part of a very classical tool in operator theory named **GNS-construction** for Israel Gel'fand, Mark Naimark, and Irving Segal.

Proof. of Claim
Let us fix $h \in \mathbb{R}[\underline{X}]$ and $d \in \mathbb{N}$. Take p and q as in (3.1) and define $|\underline{X}|^2 := X_1^2 + \cdots + X_n^2$. Since $(1+q)|\underline{X}|^{2d-2}h^2 \in T_S$ and L is non-negative on elements of T_S, we have that:

$$\begin{aligned} L(|\underline{X}|^{2d}h^2p) &\leq L(|\underline{X}|^{2d}h^2p) + L\left((1+q)|\underline{X}|^{2d-2}h^2\right) \\ &= L\left(|\underline{X}|^{2d-2}h^2(|\underline{X}|^2p + 1 + q)\right) \\ &\overset{(3.1)}{=} L\left(|\underline{X}|^{2d-2}h^2\sigma^2 p\right) = \sigma^2 L\left(|\underline{X}|^{2(d-1)}h^2p\right). \end{aligned}$$

Iterating, we get that

$$\forall\ d \in \mathbb{N},\ L(|\underline{X}|^{2d}h^2p) \leq \sigma^{2d}L(h^2p). \tag{3.3}$$

Fix $j \in \{1, \ldots, n\}$ and consider $\ell_j : \mathbb{R}[X_j] \to \mathbb{R}$ defined by $\ell_j(r) := L(rh^2)$, for all $r \in \mathbb{R}[X_j]$. Then ℓ_j is linear and $\ell_j(r^2) = L(r^2h^2) = L((rh)^2) \geq 0$, since by assumption L is non-negative on squares. Then, by Hamburger's Theorem 2.3.2 we have that there exists an $\mathbb{R}-$representing measure $\nu_{h,j}$ for ℓ_j. Therefore, for any $\lambda > 0$ and any $d \in \mathbb{N}$ we have

$$\begin{aligned} \int_{(-\infty,-\lambda)\cup(\lambda,+\infty)} \lambda^{2d} d\nu_{h,j} &\leq \int_{(-\infty,-\lambda)\cup(\lambda,+\infty)} X_j^{2d} d\nu_{h,j} \\ &\leq \int_{\mathbb{R}} X_j^{2d} d\nu_{h,j} = \ell_j(X_j^{2d}) = L(X_j^{2d}h^2) \\ &\leq L\left(X_j^{2d}h^2(|\underline{X}|^2p + 1 + q)\right) \\ &\overset{(3.1)}{=} L(X_j^{2d}h^2\sigma^2p) = \sigma^2 L(X_j^{2d}h^2p) \\ &\leq \sigma^2 L(|\underline{X}|^{2d}h^2p) \overset{(3.10)}{\leq} \sigma^{2+2d}L(h^2p). \end{aligned}$$

Hence, we proved that for any $\lambda > 0$ and any $d \in \mathbb{N}$ we have

$$\int_{(-\infty,-\lambda)\cup(\lambda,+\infty)} d\nu_{h,j} \leq \left(\frac{\sigma}{\lambda}\right)^{2d} \sigma^2 L(h^2p).$$

In particular, if we take $\lambda > \sigma$ and $d \to \infty$, then $\int_{(-\infty,-\lambda)\cup(\lambda,+\infty)} d\nu_{h,j} = 0$ and so that $\nu_{h,j}$ is supported in $[-\sigma, \sigma]$. Then

$$\begin{aligned} \|X_jh + N\|^2 &= L(X_j^2h^2) = \ell_j(X_j^2) = \int_{\mathbb{R}} X_j^2 d\nu_{h,j} = \int_{[-\sigma,\sigma]} X_j^2 d\nu_{h,j} \\ &\leq \sigma^2 \int_{[-\sigma,\sigma]} d\nu_{h,j} = \sigma^2\ell_j(1) = \sigma^2 L(h^2) = \sigma^2\|h + N\|^2. \end{aligned}$$

□(Claim)

For any $j \in \{1, \dots, n\}$, let us define the *multiplication operator* as follows

$$\begin{array}{rccc} W_j: & \mathbb{R}[\underline{X}]/N & \to & \mathbb{R}[\underline{X}]/N \\ & h+N & \mapsto & X_j h + N \end{array}$$

This is a well-defined operator with s.t. $\mathcal{D}(W_j) = \mathbb{R}[\underline{X}]/N$ is dense in $\mathcal{H}_L$ and
(a) $\underline{W_j \text{ is bounded}}$, since

$$\|W_j\|_{op} := \sup_{\substack{r \in \mathcal{D}(W_j) \\ r \neq o}} \frac{\|W_j r\|}{\|r\|} = \sup_{\substack{h \in \mathbb{R}[\underline{X}] \\ h \notin N}} \frac{\|X_j h + N\|}{\|h+N\|} \overset{\text{Claim}}{\leq} \sigma \sup_{\substack{h \in \mathbb{R}[\underline{X}] \\ h \notin N}} \frac{\|h+N\|}{\|h+N\|} = \sigma.$$

As $(\mathbb{R}[\underline{X}]/N, \|\cdot\|)$ is a normed space, this means that W_j is continuous.
(b) $\underline{W_j \text{ is symmetric}}$, since for any $h, r \in \mathbb{R}[\underline{X}]/N$ we have

$$\langle W_j h, r\rangle = L(X_j hr) = L(hX_j r) = \langle h, W_j r\rangle.$$

(c) $\underline{W_1, \dots, W_n \text{ are pairwise commuting}}$, since for any $j \neq k$ in $\{1, \dots, n\}$ and any $h \in \mathbb{R}[\underline{X}]$ we have

$$W_j W_k(h+N) = W_j(X_k h+N) = X_j X_k h+N = X_k X_j h+N = W_k W_j(h+N).$$

By Theorem 3.1.5 (applied for $Z = Y = \mathcal{H}_L$, $U = \mathbb{R}[\underline{X}]/N$, $\varphi = W_j$), there exists a unique bounded operator $\overline{W_j} : \mathcal{H}_L \to \mathcal{H}_L$ extending W_j and $\|\overline{W_j}\|_{op} = \|W_j\|_{op}$. Since each $\mathcal{D}(W_j)$ is dense in $\mathcal{H}_L$ and each W_j is bounded (so continuous), we have that properties (b) and (c) above hold also for $\overline{W_1}, \dots, \overline{W_n}$. Hence, $\overline{W_1}, \dots, \overline{W_n}$ are pairwise commuting bounded self-adjoint operators with $\mathcal{D}(\overline{W_j}) = \mathcal{H}_L$ for all $j \in \{1, \dots, n\}$. Then, by the Spectral Theorem 3.1.4, there exists a unique non-negative Radon measure μ such that

$$\langle (1+N), \overline{W_1}^{\alpha_1} \cdots \overline{W_n}^{\alpha_n}(1+N)\rangle = \int_{\mathbb{R}^n} \underline{X}^\alpha d\mu < \infty, \forall \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n \tag{3.4}$$

and μ is supported in $B_{\|\overline{W_1}\|_{op}}(0) \times \cdots \times B_{\|\overline{W_n}\|_{op}}(0) \overset{(a)}{\subseteq} [-\sigma, \sigma]^n =: Q$.
Since

$$\begin{aligned} \langle (1+N), \overline{W_1}^{\alpha_1} \cdots \overline{W_n}^{\alpha_n}(1+N)\rangle &= \langle (1+N), W_1{}^{\alpha_1} \cdots W_n{}^{\alpha_n}(1+N)\rangle \\ &= \langle (1+N), X_1{}^{\alpha_1} \cdots X_n{}^{\alpha_n} + N\rangle \\ &= L(X_1{}^{\alpha_1} \cdots X_n{}^{\alpha_n}) = L(\underline{X}^\alpha), \end{aligned}$$

(3.4) becomes

$$L(\underline{X}^\alpha) = \int_{\mathbb{R}^n} \underline{X}^\alpha d\mu, \forall \alpha \in \mathbb{N}_0^n.$$

Hence, the spectral measure μ is a $Q-$representing measure for L. It remains to show that μ is actually supported on K_S.

For each $i \in \{1, \ldots, n\}$ we have

$$0 \leq L(g_i h^2) = \int_Q g_i h^2 d\mu, \ \forall \ h \in \mathbb{R}[\underline{X}].$$

As Q is compact, we can apply the Stone-Weierstrass Theorem 2.3.27, we get

$$0 \leq L(g_i f^2) = \int_Q g_i f^2 d\mu, \ \forall \ f \in \mathcal{C}(Q).$$

Then

$$0 \leq L(g_i f) = \int_Q g_i f d\mu, \ \forall \ f \in \mathcal{C}(Q) \text{ s.t. } f \geq 0 \text{ on } Q$$

and so the linear functional

$$\begin{array}{rccc} \tilde{L}: & \mathcal{C}(Q) & \to & \mathbb{R} \\ & f & \mapsto & L(g_i f) \end{array}$$

is such that $\tilde{L}(f) \geq 0$ for all $f \geq 0$ on Q. Hence, by Riesz-Markov-Kakutani Theorem 2.2.5, there exists a unique non-negative Radon measure ν such that $\tilde{L}(f) = \int f d\nu$ for all $f \in \mathcal{C}(Q)$. But $\tilde{L}(f) = \int f g_i d\mu$ for all $f \in \mathcal{C}(Q)$, so the signed measure $g_i\mu$ must coincide with ν. Hence, $g_i\mu$ is a non-negative measure, which implies that the support of μ must be contained in the set of non-negativity of each g_i, i.e. μ is supported in K_S.

The uniqueness of the K_S-representing measure follows from Theorem 2.3.26 for $A = \mathbb{R}[\underline{X}]$ and $K = K_S$. □

The operator theoretical approach used in the proof of Theorem 3.2.1 can be also employed to provide an alternative proof to Corollary 2.3.18. This proof is indeed much closer to the original proof of this result due to Putinar [37].

Theorem 3.2.2. *Let $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ linear and $S := \{g_1, \ldots, g_s\} \subset \mathbb{R}[\underline{X}]$ such that the quadratic module M_S generated by S is Archimedean. Then there exists a unique K_S-representing measure for L if and only if $L(h^2 g_i) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$ and $i \in \{0, 1, \ldots, s\}$, where $g_0 := 1$.*

Proof. Suppose there exists a K_S-representing measure μ for L, then for any $h \in \mathbb{R}[\underline{X}]$ and any $i \in \{0, 1, \ldots, s\}$ we have

$$L(h^2 g_i) = \int_{K_S} h^2 g_i d\mu,$$

which is non-negative as integral of a non-negative function w.r.t. a non-negative measure.

Conversely, suppose that $L(h^2 g_i)$ for all $h \in \mathbb{R}[\underline{X}]$ and all $i \in \{0, 1, \dots, s\}$, i.e. $L(M_S) \geq 0$. Since $g_0 := 1$, we have that $L(h^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$. Then we can run the GNS-construction as in the proof of Theorem 3.2.1 and construct the Hilbert space $\mathcal{H}_L$ associated to L by taking the completion of $\mathbb{R}[\underline{X}]/N$ w.r.t. the inner product $\langle \cdot, \cdot \rangle$ defined in (3.2), where $N := \{f \in \mathbb{R}[\underline{X}] : L(f^2) = 0\}$. Denote by $\| \cdot \|$ the norm on $\mathcal{H}_L$ induced by $\langle \cdot, \cdot \rangle$.

In the proof of Theorem 3.2.1 the compactness of K_S and the non-negativity of L on T_S implied the following bound

$$\forall\ h \in \mathbb{R}[\underline{X}],\ j \in \{1, \dots, n\}, \quad \|X_j h + N\| \leq \sigma \|h + N\| \text{ for some } \sigma > 0, \quad (3.5)$$

which was fundamental in the rest of the proof. Here we still have compactness of K_S as M_S is Archimedean by Remark 1.3.32-c), but we have the non-negativity of L only on M_S which is contained in T_S. However, we can still derive (3.5) exploiting the Archimedeanity of M_S. Indeed, as M_S is Archimedean, for any $j \in \{1, \dots, n\}$ there exists $\lambda_j \in \mathbb{N}$ such that $\lambda_j \pm X_j^2 \in M_S$. This together with the non-negativity of L on M_S gives in particular that $L(h^2(\lambda_j - X_j^2)) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$. Hence, for each $j \in \{1, \dots, n\}$ and for each $h \in \mathbb{R}[\underline{X}]$, we obtain

$$\|X_j h + N\|^2 = L(X_j^2 h^2) \leq L(\lambda_j h^2) = \lambda_j L(h^2) \leq \sigma^2 \|h + N\|,$$

where $\sigma^2 := \max_{j=1,\dots,n} \lambda_j$. This proves that (3.5) holds and so we can continue the proof exactly as in the proof of Theorem 3.2.1 and show that there exists a K_S-representing measure. As for the uniqueness, we can apply also here Theorem 2.3.26 for $A = \mathbb{R}[\underline{X}]$ and $K = K_S$ since the Archimedeanity of M_S ensures that K_S is compact. □

3.3 Solving the KMP for K non-compact semialgebraic sets

Having in mind Theorem 3.2.1 and Theorem 3.2.2, it is natural to ask if the non-negativity of a linear functional on T_S or M_S is still sufficient to get the existence of a K_S-representing measure when K_S is not compact (and so M_S is not Archimedean). We already know that this is true for $K_S \subseteq \mathbb{R}$ with $S \supseteq S_{nat}$ by Corollary 2.3.1 (see also Theorems 2.3.2 and 2.3.3). But what about higher dimensions? In this section, we are going to see how the operator theoretical approach to the KMP sheds some light on this question.

A crucial role will be played by the following condition which will be further discussed in the next chapter.

Definition 3.3.1. *Given a sequence $m := (m_\alpha)_{\alpha\in\mathbb{N}_0^n}$ of non-negative real numbers, we say that m fulfills the* Carleman condition *if*

$$\sum_{k=1}^{\infty} m_{(0,\dots0,\underbrace{2k}_{j\text{-}th},0,\dots,0)}^{-\frac{1}{2k}} = \infty, \quad \forall j \in \{1,\dots,n\}. \tag{3.6}$$

Let us start by a result due to Nussbaum, who obtained in [35, Theorem 10] a solution to the KMP for $K = \mathbb{R}^n$ as a consequence of an important result concerning the theory of unbounded operators, namely Theorem 3.1.10. Indeed, in this case the multiplication operators defined in the previous section are not anymore guaranteed to be bounded, because we do not have either compactness or Archimedianity to ensure that the bound (3.5) holds. Hence, we need to deal with unbounded operators and use the results in Section 3.1.2.

Theorem 3.3.2.
Let $n \geq 2$ be an integer and $L : \mathbb{R}[X_1,\dots,X_n] \to \mathbb{R}$ linear. If $L(h^2) \geq 0$ for all $h \in \mathbb{R}[X_1,\dots,X_n]$ and fulfills the Carleman condition, i.e.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{L(X_j^{2k})}} = \infty, \quad \forall j \in \{1,\dots,n\}, \tag{3.7}$$

then there exists a unique $\mathbb{R}^n$−representing measure for L. Conversely, if there exists a unique $\mathbb{R}^n$−representing measure for L then $L(h^2) \geq 0$ for all $h \in \mathbb{R}[X_1,\dots,X_n]$.

The existence part of this theorem is a higher dimensional version of Hamburger's theorem 2.3.2. We provide a proof just for the case $n = 2$, since the proof structure for $n \geq 3$ is exactly the same. Afterwards, we will see how this proof can be adapted to the case $n = 1$, giving an alternative proof to Hamburger's theorem 2.3.2.

Proof. of Existence in Theorem 3.3.2 for $n = 2$.
Suppose there exists a $\mathbb{R}^2$−representing measure μ for L, then for any polynomial $h \in \mathbb{R}[X_1,X_2] =: \mathbb{R}[\underline{X}]$ we have $L(h^2) = \int_{\mathbb{R}^2} h^2 d\mu$, which is non-negative as integral of a non-negative function w.r.t. a non-negative measure.

Conversely, suppose that $L(h^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$ and that the Carleman condition (3.7) holds. Then we can run the GNS-construction as in the previous section and construct the Hilbert space $\mathcal{H}_L$ associated to L by taking the completion of $\mathbb{R}[\underline{X}]/N$ w.r.t. the inner product $\langle\cdot,\cdot\rangle$ defined in (3.2), where $N := \{f \in \mathbb{R}[\underline{X}] : L(f^2) = 0\}$. Denote by $\|\cdot\|$ the norm on $\mathcal{H}_L$ induced by $\langle\cdot,\cdot\rangle$. For any $j \in \{1,2\}$, let us define the *multiplication operator* as follows

$$\begin{array}{rrcl} W_j: & \mathbb{R}[\underline{X}]/N & \to & \mathbb{R}[\underline{X}]/N \\ & h+N & \mapsto & X_j h + N \end{array}$$

This is a well-defined operator which is densely defined in $\mathcal{H}_L$ and symmetric, since $\mathcal{D}(W_j) = \mathbb{R}[\underline{X}]/N$ and

$$\langle W_j h, r\rangle = L(X_j hr) = L(hX_j r) = \langle h, W_j r\rangle, \quad \forall\, h,r \in \mathbb{R}[\underline{X}]/N.$$

Since the multiplication operators are unbounded, we aim to use the Spectral Theorem 3.1.8 and so we need to find pairwise strongly commuting self-adjoint extensions of the multiplication operators in $\mathcal{H}_L$. To this purpose, let us consider the set

$$\mathcal{D} := \{X_1^s X_2^t + N | s,t \in \mathbb{N}_0\}$$

and show that W_1, W_2 and $\mathcal{D}$ fulfill all the assumptions of Theorem 3.1.10.

a) $W_1\mathcal{D} \subset \mathcal{D}$ and $W_2\mathcal{D} \subset \mathcal{D}$ directly follow from the definitions of W_1, W_2 and $\mathcal{D}$.

b) For all $h \in \mathcal{D}$, say $h = X_1^s X_2^t + N$ for some $s,t \in \mathbb{N}_0$, we have

$$W_1W_2(h+N) = X_1^{s+1}X_2^{t+1} + N = X_2^{t+1}X_1^{s+1} + N = W_2W_1(h+N).$$

c) $\mathcal{D}$ is total in $\mathcal{H}_L$ since $\mathrm{span}(\mathcal{D}) = \mathbb{R}[\underline{X}]/N$ which is dense in $\mathcal{H}_L$ by construction.

d) *Claim*: Any $h \in \mathcal{D}$ is a quasi-analytic vector for both W_1 and W_2.

Then Theorem 3.1.10 guarantees that there exist unique self-adjoint extensions $\overline{W_1}$ and $\overline{W_2}$ of W_1 and W_2 in $\mathcal{H}_L$ s.t. $\overline{W_1}$ and $\overline{W_2}$ are strongly commuting. Moreover, $1 + N \in \mathcal{D}(W_1) = \mathcal{D}(W_2) = \mathbb{R}[\underline{X}]/N \subset \mathcal{H}_L$ is s.t. $\forall d \in \mathbb{N}_0$, $\forall\, i_1, \ldots, i_{d+1} \in \{1,2\}$ we have

$$\overline{W_{i_d}} \cdot \overline{W_{i_{d-1}}} \cdots \overline{W_{i_1}}(1+N) = X_{i_d} \cdots X_{i_1} + N \in \mathcal{D}(W_{i_{d+1}}) = \mathbb{R}[\underline{X}]/N.$$

Then we can apply the Spectral Theorem 3.1.8 to $\overline{W_1}$ and $\overline{W_2}$ and get that there exists a unique non-negative Radon measure μ on $\mathbb{R}^2$ such that

$$\langle (1+N), \underbrace{\overline{W_1}\cdots\overline{W_1}}_{\alpha_1 \text{ times}}\underbrace{\overline{W_2}\cdots\overline{W_2}}_{\alpha_2 \text{ times}}\cdot(1+N)\rangle = \int_{\mathbb{R}^2} X_1^{\alpha_1}X_2^{\alpha_2}d\mu(X_1,X_2),\ \forall\, \alpha_1,\alpha_2 \in \mathbb{N}_0. \tag{3.8}$$

Since

$$\begin{aligned}\langle (1+N), \overline{W_1}^{\alpha_1}\overline{W_2}^{\alpha_2}(1+N)\rangle &= \langle (1+N), W_1{}^{\alpha_1}W_2{}^{\alpha_2}(1+N)\rangle \\ &= \langle (1+N), X_1{}^{\alpha_1}X_2{}^{\alpha_2}+N\rangle \\ &= L(X_1{}^{\alpha_1}X_2{}^{\alpha_2}) = L(\underline{X}^{\alpha}),\end{aligned}$$

(3.8) becomes $L(\underline{X}^{\alpha}) = \int_{\mathbb{R}^2} \underline{X}^{\alpha} d\mu, \forall \alpha \in \mathbb{N}_0^2$. Hence, the spectral measure μ is an $\mathbb{R}^2-$representing measure for L. $\square$