Since

$$\begin{aligned}\langle(1+N),\overline{W_1}^{\alpha_1}\overline{W_2}^{\alpha_2}(1+N)\rangle &= \langle(1+N),W_1^{\alpha_1}W_2^{\alpha_2}(1+N)\rangle\\ &= \langle(1+N),X_1^{\alpha_1}X_2^{\alpha_2}+N\rangle\\ &= L(X_1^{\alpha_1}X_2^{\alpha_2})=L(\underline{X}^{\alpha}),\end{aligned}$$

(3.8) becomes $L(\underline{X}^{\alpha})=\int_{\mathbb{R}^2}\underline{X}^{\alpha}d\mu, \forall\alpha\in\mathbb{N}_0^2$. Hence, the spectral measure μ is an $\mathbb{R}^2-$representing measure for L. □

Note that the fact that μ is the unique spectral measure coming from the unique self-adjoint extensions of the multiplication operators to $\mathcal{H}_L$ does not guarantee that μ is the unique $\mathbb{R}^n-$representing measure for L. Indeed, there could exist self-adjoint extension (resp. pairwise strongly commuting extensions) of the multiplication operators in another Hilbert space larger than $\mathcal{H}_L$ such that the corresponding spectral measure ν is also an $\mathbb{R}^n-$representing for L but clearly does not coincide with μ. Hence, we need an extra argument to show the uniqueness of the representing measure.

Before passing to the determinacy part, let us complete the existence part by showing that the Claim d) holds. To do that we will need the notion of log-convex sequences and some of their properties.

Definition 3.3.3.
A sequence $(s_k)_{k\in\mathbb{N}_0}$ of non-negative real numbers is said to be log-convex *if for all $k\in\mathbb{N}$ we have that $s_k^2\leq s_{k-1}s_{k+1}$.*

Lemma 3.3.4. *A sequence $(s_k)_{k\in\mathbb{N}_0}$ of positive real numbers is log-convex if and only if $\left(\sqrt[k]{\frac{s_k}{s_0}}\right)_{k\in\mathbb{N}}$ is monotone increasing.*

Proof.
The log-convexity of $(s_k)_{k\in\mathbb{N}_0}$ is equivalent to the sequence $\left(\frac{s_k}{s_{k-1}}\right)_{k\in\mathbb{N}}$ being increasing, since for any $k\in\mathbb{N}$ we have that

$$s_k^2\leq s_{k-1}s_{k+1}\Leftrightarrow\frac{s_k}{s_{k-1}}\leq\frac{s_{k+1}}{s_k}.$$

Hence, for any $k\in\mathbb{N}$ we get

$$\frac{s_k}{s_0}=\prod_{j=1}^{k}\frac{s_j}{s_{j-1}}\leq\left(\frac{s_k}{s_{k-1}}\right)^k,$$

i.e. $s_{k-1}^k\leq s_0s_k^{k-1}$. By multiplying the latter on both sides by $\frac{1}{s_0^k}$ we get $\left(\frac{s_{k-1}}{s_0}\right)^k\leq\left(\frac{s_k}{s_0}\right)^{k-1}$, which is equivalent to $\left(\frac{s_{k-1}}{s_0}\right)^{\frac{1}{k-1}}\leq\left(\frac{s_k}{s_0}\right)^{\frac{1}{k}}$. □

Lemma 3.3.5. *Let $(s_k)_{k\in\mathbb{N}_0}$ be a sequence of non-negative real numbers s.t. $s_{2k} > 0$ for all $k \in \mathbb{N}_0$ and $(s_{2k})_{k\in\mathbb{N}_0}$ is log-convex. Then*

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{s_{2k}}} = \infty \Longleftrightarrow \sum_{k=1}^{\infty} \frac{1}{\sqrt[4k]{s_{4k}}} = \infty.$$

Proof. (see Bonus Sheet) □

Lemma 3.3.6.
Let $o \neq q, f \in \mathbb{R}[\underline{X}]$ and $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ linear s.t. $L(h^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$. Define $\tilde{L}_f : \mathbb{R}[\underline{X}] \to \mathbb{R}$ as $\tilde{L}_f(p) := L(fp)$ for all $p \in \mathbb{R}[\underline{X}]$. Then

$$\left(\tilde{L}_f(h^2) \geq 0,\ \forall h \in \mathbb{R}[\underline{X}] \text{ and } \sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{L(q^{2k})}} = \infty\right) \Longrightarrow \sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{\tilde{L}_f(q^{2k})}} = \infty.$$

Proof.
For any $k \in \mathbb{N}_0$, set $t_k := L(q^k)$ and $r_k := \tilde{L}_f(q^k)$. Since $L(h^2) \geq 0$ and $\tilde{L}_f(h^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$, we have that $t_{2k} \geq 0$, $r_{2k} \geq 0$ for all $k \in \mathbb{N}_0$ and we can apply the Cauchy-Schwarz inequality to both L and $\tilde{L}_f$. Hence, we obtain that the following hold for all $k \in \mathbb{N}_0$

$$t_{2k+2}^2 = \left(L(q^{2k+2})\right)^2 = \left(L(q^k q^{k+2})\right)^2 \leq L(q^{2k})L(q^{2k+4}) = t_{2k}t_{2k+4} \tag{3.9}$$

$$r_{2k}^2 = \left(L(fq^{2k})\right)^2 \leq L(q^{4k})L(f^2) = t_{4k}L(f^2) \tag{3.10}$$

Now w.l.o.g. we can assume that $t_{2k} > 0$ for all $k \in \mathbb{N}_0$ and $L(f^2) > 0$. Indeed,

- If $t_{2j} = 0$ for some $j \in \mathbb{N}_0$, then by (3.9) we have that $t_{2k} = 0$ for all $k \geq j$ in $\mathbb{N}_0$ and so by (3.10) also $r_{2k} = 0$ for all $k \geq j$ in $\mathbb{N}_0$. Hence, $\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{r_{2k}}} = \infty$ and we have already our desired conclusion.
- If $L(f^2) = 0$, then $r_{2k} = 0$ for all $k \in \mathbb{N}_0$ and so again our desired conclusion holds.

Hence, $(t_{2k})_{k\in\mathbb{N}_0}$ is a sequence of positive real numbers, which is log-convex by (3.9). Since by assumption $\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{t_{2k}}} = \infty$, we can apply Lemma 3.3.5 and obtain that

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[4k]{t_{4k}}} = \infty \tag{3.11}$$

Therefore, we get

$$r_{2k}^{-\frac{1}{2k}} \overset{(3.10)}{\geq} t_{4k}^{-\frac{1}{4k}} \left(L(f^2)\right)^{-\frac{1}{4k}} \geq c_f t_{4k}^{-\frac{1}{4k}},\ \forall\, k \in \mathbb{N}, \tag{3.12}$$

where $c_f := \left(1 + L(f^2)\right)^{-1}$ is clearly a positive constant. Then

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{r_{2k}}} \overset{(3.12)}{\geq} c_f \sum_{k=1}^{\infty} \frac{1}{\sqrt[4k]{t_{4k}}} \overset{(3.11)}{=} \infty.$$

□

Corollary 3.3.7. *Let $o \neq f \in \mathbb{R}[\underline{X}]$ and $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ linear s.t. $L(h^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$. Suppose that $\tilde{L}_f(h^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$. If L fulfills Carleman condition (3.7), then so does $\tilde{L}_f$.*

Proof. Apply Lemma 3.3.6 for $q = X_j$ for each $j \in \{1, \dots, n\}$. □

Proof. of Claim d).
Let us fix $s, t \in \mathbb{N}_0$, then by using the Cauchy-Schwarz inequality we get that for any $k \in \mathbb{N}$

$$\left\| {W_1}^k X_1^s X_2^t \right\|^2 = \left(L(X_1^{2(k+s)} X_2^{2t}) \right) \leq \left(L(X_1^{4(k+s)}) \right)^{\frac{1}{2}} \left(L(X_2^{4t}) \right)^{\frac{1}{2}}$$

which gives in turn that

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{\left\| {W_1}^k X_1^s X_2^t \right\|}} \geq \sum_{k=1}^{\infty} \frac{1}{\sqrt[4k]{L(X_1^{4(k+s)}) L(X_2^{4t})}}. \tag{3.13}$$

W.l.o.g. we can assume that $c := L(X_2^{4t}) > 0$ and that for any $k \in \mathbb{N}$ we have $L(X_1^{4(k+s)}) > 0$ (otherwise the series on the right-hand side of (3.13) would diverge and we would have already our conclusion).

Then $L(cX_1^{4s} h^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$. This together with (3.7) ensures that we can apply Lemma 3.3.6 for $q := X_1$ and $f := cX_1^{4s}$ obtaining that $\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{cL(X_1^{2k+4s})}} = \infty$.

Moreover, the sequence $\left(L(cX_1^{2k+4s}) \right)_{k \in \mathbb{N}_0}$ is log-convex (see Definition 3.3.3), since for any $k \in \mathbb{N}$ we have

$$\left[L(cX_1^{2k+4s}) \right]^2 = \left[L(\sqrt{c} X_1^{k-1+2s} \sqrt{c} X_1^{k+1+2s}) \right]^2 \leq L(cX_1^{(2k-2)+4s}) L(cX_1^{(2k+2)+4s}).$$

Then we have that

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[4k]{cL(X_1^{4k+4s})}} = \infty. \tag{3.14}$$

In fact, we can distinguish two cases:

- If there exists $w \in \mathbb{N}_0$ such that $L(cX_1^{2w+4s}) = 0$, then by log-convexity $L(cX_1^{2k+4s}) = 0$ for all integers $k \geq w$, which implies that (3.14) holds.
- if $L(cX_1^{2k+4s}) > 0$ for all $k \in \mathbb{N}$, then by Lemma 3.3.5 we have that (3.14) holds.

Hence, (3.14) and (3.13) guarantee that $\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{\| |W_1{}^k X_1^s X_2^t\| |}} = \infty$, i.e. $X_1^s X_2^t$ is a quasi-analytic vector for W_1.

The same proof applies to show that $X_1^s X_2^t$ is a quasi-analytic vector for W_2. □(Claim d))

The proof of the existence part of Theorem 3.3.2 can be adapted to provide an alternative proof to Hamburger's theorem 2.3.2. Note that for $n = 1$ the Carleman condition is not needed for getting the existence of an $\mathbb{R}^n-$representing measure for L, while this was essential for getting it in the case $n \geq 2$. We will see that Carleman's condition is instead crucial in proving the determinacy of the $\mathbb{R}^n-$representing measure independently of the dimension n.

Theorem 3.3.8. *Let $L : \mathbb{R}[X] \to \mathbb{R}$ be linear. There exists an $\mathbb{R}-$representing measure for L if and only if $L(h^2) \geq 0$ for all $h \in \mathbb{R}[X]$.*
If in addition, L fulfills the Carleman condition (3.7) for $n = 1$, i.e.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{L(X^{2k})}} = \infty. \tag{3.15}$$

then the representing measure is determinate.

Proof. of Existence in Theorem 3.3.8, i.e. of Hamburger's theorem 2.3.2
Suppose there exists an $\mathbb{R}-$representing measure μ for L, then for any polynomial $h \in \mathbb{R}[X]$ we have $L(h^2) = \int_{\mathbb{R}} h^2 d\mu$, which is non-negative as integral of a non-negative function w.r.t. a non-negative measure.

Conversely, suppose that $L(h^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$. Then we can run the GNS-construction and construct the Hilbert space $\mathcal{H}_L$ associated to L. Consider the multiplication operator

$$\begin{array}{rccc} W: & \mathbb{R}[X]/N & \to & \mathbb{R}[X]/N \\ & h+N & \mapsto & X_j h + N \end{array}$$

where $N := \{f \in \mathbb{R}[X] : L(f^2) = 0\}$. Since W is a symmetric unbounded operator densely defined in $\mathcal{H}_L$, it admits a self-adjoint extension $\overline{W}$ in $\mathcal{H}_L$ (see e.g. [39, p.319]). Then by the Spectral Theorem 3.1.8 for $n = 1$ and

$v = 1 + N \in \mathcal{D}^\infty(W) = \mathbb{R}[\underline{X}]/N \subset \mathcal{H}_L$ we get that there exists a unique non-negative Radon measure μ on $\mathbb{R}$ such that

$$\langle (1+N), \overline{W}^j(1+N) \rangle = \int_{\mathbb{R}} X^j d\mu(X), \ \forall \ j \in \mathbb{N}_0. \tag{3.16}$$

Since

$$\langle (1+N), \overline{W}^j(1+N) \rangle = \langle (1+N), W^j(1+N) \rangle = \langle (1+N), X^j + N \rangle = L(X^j)$$

(3.16) becomes $L(X) = \int_{\mathbb{R}} X^j d\mu(X), \forall \ j \in \mathbb{N}_0$. Hence, the spectral measure μ is an $\mathbb{R}-$representing measure for L. □

Let us show now the determinacy part of both Theorem 3.3.2 and Theorem 3.3.8. This will be a consequence of the following important result about the determinacy of the moment problem, which we are going to prove in the next chapter.

Theorem 3.3.9.
Let $n \in \mathbb{N}$. If μ is a non-negative Radon measure on $\mathbb{R}^n$ such that the sequence of its moments $(m_\alpha^\mu)_{\alpha \in \mathbb{N}_0^n}$ exists and fulfills the Carleman condition (3.6), then μ is determinate, i.e. any other non-negative Radon measure having the same moment sequence as μ must coincide with μ.

Proof. (of Uniqueness in Theorem 3.3.2 and in Theorem 3.3.8)
Let μ, ν be two $\mathbb{R}^n-$representing measure for L. Then μ and ν have the same moment sequence $(L(\underline{X}^\alpha))_{\alpha \in \mathbb{N}_0^n}$. Since by assumption L fulfills (3.7), the sequence $(L(\underline{X}^\alpha))_{\alpha \in \mathbb{N}_0^n}$ fulfills (3.6) and so Theorem 3.3.9 ensures that $\mu = \nu$. □

Carleman's condition, and so Theorem 3.3.9, will also play a crucial role to prove a version of Theorem 3.3.2 for the KMP with K (not necessarily compact) b.c.s.a.s. of $\mathbb{R}^n$ due to Lasserre [28, Theorem 3.2] (see also [19, Theorem 5.1]).

Theorem 3.3.10.
Let $n, s \in \mathbb{N}$, $S := \{g_1, \dots, g_s\} \subset \mathbb{R}[X_1, \dots, X_n]$, and $L : \mathbb{R}[X_1, \dots, X_n] \to \mathbb{R}$ linear s.t. $L(h^2) \geq 0$ for all $h \in \mathbb{R}[X_1, \dots, X_n]$ and Carleman's condition (3.7) holds. Then there exists a unique K_S-representing measure for L if and only if $L(g_i h^2) \geq 0$ for all $h \in \mathbb{R}[X_1, \dots, X_n]$ and all $i \in \{1, \dots, s\}$.