Proof.
Since $L(h^2) \geq 0$ for all $h \in \mathbb{R}[X_1, \ldots, X_n] =: \mathbb{R}[\underline{X}]$ and L fulfills the Carleman condition (3.7), Theorem 3.3.2 guarantees that there exists a unique $\mathbb{R}^n-$representing measure μ for L. We want to show that μ is actually supported on K_S.
Case $s = 1$
For notational convenience, let us first consider the case $s = 1$ and so $S := \{g\}$. Define $\tilde{L}_g : \mathbb{R}[\underline{X}] \to \mathbb{R}$ as $\tilde{L}_g(p) := L(pg)$ for all $p \in \mathbb{R}[\underline{X}]$. Since $\tilde{L}_g(h^2) = L(gh^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$ and L satisfies the Carleman condition (3.7), Lemma 3.3.6 (applied for $q = X_j$ with $j = 1, \ldots, n$ and $f = g$) ensures that $\tilde{L}_g$ also fulfils Carleman's condition. Hence, by applying again Theorem 3.3.2 we get that there exists a unique $\mathbb{R}^n-$representing measure η for $\tilde{L}_g$. Thus, we obtained that

$$\int_{\mathbb{R}^n} \underline{X}^\alpha d\eta(\underline{X}) = \tilde{L}_g(\underline{X}^\alpha) = L(g\underline{X}^\alpha) = \int_{\mathbb{R}^n} \underline{X}^\alpha \underbrace{g(\underline{X})d\mu(\underline{X})}_{=:d\nu(\underline{X})}, \ \forall\, \alpha \in \mathbb{N}_0^n. \quad (3.17)$$

The measure ν is a signed Radon measure on the Borel $\sigma-$algebra $\mathcal{B}(\mathbb{R}^n)$ on $\mathbb{R}^n$ and can be written as $\nu = \nu_+ - \nu_-$, where

$$\begin{aligned} d\nu_+ &:= \ \mathbb{1}_{\Gamma^+} d\nu \quad \text{with} \quad \Gamma^+ := \{x \in \mathbb{R}^n : g(x) \geq 0\} \\ d\nu_- &:= -\mathbb{1}_{\Gamma^-} d\nu \quad \text{with} \quad \Gamma^- := \{x \in \mathbb{R}^n : g(x) < 0\} \end{aligned}$$

and so ν_+ and ν_- are both non-negative Radon measures on $\mathbb{R}^n$.

Claim: $\nu_- \equiv 0$.

Proof.
Define the following two non-negative Radon measures on $\mathcal{B}(\mathbb{R}^n)$

$$d\mu_+ := \mathbb{1}_{\Gamma^+} d\mu \text{ and } d\mu_- := \mathbb{1}_{\Gamma^-} d\mu.$$

Then $\mu = \mu_+ + \mu-$ and so we have

$$\int_{\mathbb{R}^n} X_j^{2k} d\mu_+(\underline{X}) \leq \int_{\mathbb{R}^n} X_j^{2k} d\mu(\underline{X}), \ \forall\, k \in \mathbb{N}_0, \forall j = 1, \ldots, n. \quad (3.18)$$

Consider $\ell_{\mu_+} : \mathbb{R}[\underline{X}] \to \mathbb{R}$ defined by $\ell_{\mu_+}(p) := \int_{\mathbb{R}^n} p d\mu_+$. Then (3.18) can be rewritten as

$$\ell_{\mu_+}(X_i^{2k}) \leq L(X_j^{2k}), \ \ \forall\, k \in \mathbb{N}_0, \forall j = 1, \ldots, n,$$

which implies that

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{\ell_{\mu_+}(X_i^{2k})}} \geq \sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{L(X_j^{2k})}} \overset{\text{hp}}{=} \infty,\ \forall\, k \in \mathbb{N}_0, \forall j = 1, \ldots, n,$$

i.e. ℓ_{μ_+} fulfills the Carleman condition.

Consider $\ell_{\nu_+} : \mathbb{R}[\underline{X}] \to \mathbb{R}$ defined by $\ell_{\nu_+}(p) := \int_{\mathbb{R}^n} p d\nu_+$. Then

$$\ell_{\nu_+}(p) = \int_{\mathbb{R}^n} p \mathbb{1}_{\Gamma^+} g d\mu = \int_{\mathbb{R}^n} pg d\mu_+ = \ell_{\mu_+}(pg),\ \forall p \in \mathbb{R}[\underline{X}]$$

and

$$\ell_{\nu_+}(h^2) = \int_{\mathbb{R}^n} h^2 d\nu_+ \geq 0\ \forall h \in \mathbb{R}[\underline{X}].$$

Hence, by Lemma 3.3.6 (applied for $L = \ell_{\mu_+}$, $q = X_j$,$f = g$), we get that also ℓ_{ν_+} fulfills the Carleman condition and so that ν_+ is determinate by Theorem 3.3.9.

Putting all together, we obtain that for all $\alpha \in \mathbb{N}_0^n$

$$\begin{aligned}
\int_{\mathbb{R}^n} \underline{X}^\alpha d\nu_+(\underline{X}) &\overset{\text{def}}{=} \int_{\mathbb{R}^n} \underline{X}^\alpha g(\underline{X}) d\mu_+(\underline{X}) \\
&\overset{\mu=\mu_+\mu_-}{=} \int_{\mathbb{R}^n} \underline{X}^\alpha g(\underline{X}) d\mu(\underline{X}) - \int_{\mathbb{R}^n} \underline{X}^\alpha g(\underline{X}) d\mu_-(\underline{X}) \\
&\overset{(3.17)}{=} \int_{\mathbb{R}^n} \underline{X}^\alpha d\eta(\underline{X}) - \int_{\mathbb{R}^n} \underline{X}^\alpha g(\underline{X}) d\mu_-(\underline{X}) \\
&\overset{\text{def}}{=} \int_{\mathbb{R}^n} \underline{X}^\alpha d\eta(\underline{X}) + \int_{\mathbb{R}^n} \underline{X}^\alpha d\nu_-(\underline{X}) \\
&= \int_{\mathbb{R}^n} \underline{X}^\alpha d(\eta + \nu_-)(\underline{X}),
\end{aligned}$$

i.e. the non-negative Radon measures ν_+ and $\eta + \nu_-$ have the same moments. Since ν_+ is determinate, they need to coincide, i.e. $\nu_+ \equiv \eta + \nu_-$. Hence, for any $B \in \mathcal{B}(\mathbb{R}^n)$ we have $0 = \nu_+(\Gamma^-) \geq \nu_-(\Gamma^-) \geq 0$, that is, $\nu_-(\Gamma^-) = 0$. Since by definition $\nu_-(\Gamma^+) = 0$ and $\mathbb{R}^n = \Gamma^+ \cup \Gamma^-$, we get that $\nu_- \equiv 0$. $\square$(Claim)

The Claim implies that μ is supported on Γ^+, i.e. for any $B \in \mathcal{B}(\mathbb{R}^n)$ such that $B \cap \Gamma^+ = \emptyset$ we have $\mu(B) = 0$. In fact, suppose that this is not the case. Then there exists $\varepsilon > 0$ such that $\overline{B}_\varepsilon \cap \Gamma^+ = \emptyset$ but $\mu\left(\overline{B}_\varepsilon\right) > 0$, where $\overline{B}_\varepsilon$ is some closed ball in $\mathbb{R}^n$ of radius ε. Then for any $x \in \overline{B}_\varepsilon$ we have that $x \in \Gamma^-$ and so $g(x) < 0$, i.e. $-g(x) > 0$. Hence, we get

$$0 \overset{\text{Claim}}{=} \nu_-(\overline{B}_\varepsilon) = \int_{\overline{B}_\varepsilon} -\mathbb{1}_{\Gamma^-} d\nu = \int_{\overline{B}_\varepsilon} -g(\underline{X}) d\mu(\underline{X}) \geq \left(\min_{x \in \overline{B}_\varepsilon} -g(x)\right) \mu(\overline{B}_\varepsilon) > 0,$$

which yields a contradiction.

Thus, we proved that μ is supported on $\{x \in \mathbb{R}^n : g(x) \geq 0\}$, which in this case coincides with K_S.

Case $s \geq 2$

Suppose now that $s > 1$ and $S := \{g_1, \ldots, g_s\}$. By repeating for each g_i the same proof as above, we get that μ is supported on each $\{x \in \mathbb{R}^n : g_i(x) \geq 0\}$ with $i \in \{1, \ldots, s\}$. Hence, we get that

$$\begin{aligned} 0 \leq \mu\left(\mathbb{R}^n \setminus K_S\right) &= \mu\left(\bigcup_{i=1}^{s} \mathbb{R}^n \setminus \{x \in \mathbb{R}^n : g_i(x) \geq 0\}\right) \\ &\leq \sum_{i=1}^{s} \mu\left(\mathbb{R}^n \setminus \{x \in \mathbb{R}^n : g_i(x) \geq 0\}\right) = 0, \end{aligned}$$

i.e. μ is supported on K_S.

□

Chapter 4

Determinacy of the $K-$Moment Problem

In this chapter we are going to investigate the so-called *determinacy question*, which is certainly one of the most investigated aspects of the $K-$moment problem. The determinacy question consists in finding under which conditions a non-negative measure with given support K is completely determined by its moments. In particular, we will see how the concept of quasi-analyticity enters in the study of the determinacy question and give a proof of Theorem 3.3.9 first for $n = 1$ and then for higher dimensions.

From now on, for $K \subseteq \mathbb{R}^n$ closed, we denote by $\mathcal{M}^*(K)$ the collection of all the non-negative Radon measures on $\mathbb{R}^n$ having finite moments of all orders and which are supported in K.

Definition 4.0.1. *A measure $\mu \in \mathcal{M}^*(K)$ is said to be $K-$*determinate *if for any $\nu \in \mathcal{M}^*(K)$ such that $\int x^\alpha d\mu(x) = \int x^\alpha d\nu(x), \forall\ \alpha \in \mathbb{N}_0^n$ we have that $\mu \equiv \nu$. Equivalently a sequence of real numbers m (resp. a linear functional L on $\mathbb{R}[\underline{X}]$) is called $K-$determinate if there exists at most one $K-$representing measure for m (resp. for L).*

Note that if K_1 and K_2 are closed subsets of $\mathbb{R}^n$ such that $K_1 \subset K_2$, then the K_2-determinacy always implies the K_1-determinacy but the converse does not hold in general.

4.1 Quasi-analytic classes

Let us recall the basic definitions and state some preliminary results concerning the theory of quasi-analytic functions. In the following, we denote by $\mathcal{C}^\infty(X)$ the space of all infinitely differentiable real valued functions defined on a topological space X.

Definition 4.1.1.
Given a sequence of positive real numbers $(s_j)_{j\in\mathbb{N}_0}$ *and an open* $I \subseteq \mathbb{R}$*, we define the class* $C\{s_j\}$ *as the set of all functions* $f \in \mathcal{C}^\infty(I)$ *for which there exists* $\gamma_f > 0$ *(only depending on* f*) such that* $\left\|D^k f\right\|_\infty \leq (\gamma_f)^k s_k, \quad \forall\, k \in \mathbb{N}_0$*, where* $D^k f$ *is the* $k-th$ *derivative of* f *and* $\left\|D^k f\right\|_\infty := \sup_{x\in I}\left|D^k f(x)\right|$*.*

The class $C\{s_j\}$ *of functions on* I *is said to be* quasi-analytic *if the conditions*

$$f \in C\{s_j\},\ \exists\ t_0 \in I s.t.\ \ (D^k f)(t_0) = 0, \quad \forall\, k \in \mathbb{N}_0$$

imply that $f(x) = 0$ *for all* $x \in I$*.*

The problem to give necessary and sufficient conditions bearing on the sequence $(s_j)_{j\in\mathbb{N}_0}$ such that the class $C\{s_j\}$ is quasi-analytic was proposed by Hadamard in [17]. Denjoy was the first to provide sufficient conditions for the quasi-analyticity of a class [10], but the problem was completely solved by Carleman, who generalized Denjoy's theorem and methods giving the first characterization of quasi-analytic classes in [6].

Theorem 4.1.2 (The Denjoy-Carleman Theorem)**.**
Let $(s_j)_{j\in\mathbb{N}_0}$ *be a sequence of positive real numbers. The class* $C\{s_k\}$ *is quasi-analytic if and only if*

$$\sum_{k=1}^{\infty} \frac{1}{\inf_{j\geq k} \sqrt[j]{s_j}} = \infty.$$

Proof. see e.g. [8] for a simple but detailed proof. □

Corollary 4.1.3. *If* $(s_j)_{j\in\mathbb{N}_0}$ *is a sequence of positive real numbers such that*

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{s_k}} = \infty,$$

then the class $C\{s_j\}$ *is quasi-analytic.*

Proof. For any $k \in \mathbb{N}$ we have $\inf_{j\geq k} \sqrt[j]{s_j} \leq \sqrt[k]{s_k}$ and so

$$\sum_{k=1}^{\infty} \frac{1}{\inf_{j\geq k} \sqrt[j]{s_j}} \geq \sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{s_k}}.$$

Since by assumption the series on right-hand side diverges, so does the series on the left-hand side. Hence, by Theorem 4.1.2, the class $C\{s_j\}$ is quasi-analytic. □