Remark 4.1.4. *If* $(s_j)_{j\in\mathbb{N}_0}$ *is a log-convex sequence of positive real numbers such that* $s_0 = 1$, *then in Corollary 4.1.3 also the converse implication holds. Indeed, under these assumptions the sequence* $(\sqrt[j]{s_j})_{j\in\mathbb{N}}$ *is increasing by Lemma 3.3.4 and so for each* $k \in \mathbb{N}$ *we have* $\inf_{j\geq k} \sqrt[j]{s_j} = \sqrt[k]{s_k}$*.Hence, the condition* $\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{s_k}} = \infty$ *is equivalent to* $\sum_{k=1}^{\infty} \frac{1}{\inf_{j\geq k} \sqrt[j]{s_j}} = \infty$ *and so to the quasi-analiticity of the class* $C\{s_j\}$ *by Theorem 4.1.2.*

Using Corollary 4.1.3, we can easily produce some examples of quasi-analytic classes.

Examples 4.1.5.

- *The class* $\mathcal{C}\{j^j\}$ *is quasi-analytic, since* $\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{k^k}} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$.
- *The class* $\mathcal{C}\{j!\}$ *is quasi-analytic, since* $\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{k!}} \geq \sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{k^k}} = \infty$. *This is in fact the class of real analytic functions. Recall that a function* f *is real analytic on* $I \subseteq \mathbb{R}$ *if* $f \in \mathcal{C}^{\infty}(I)$ *and the Taylor series of* f *at any point* $x_0 \in I$ *pointwise converges to* f *in a neighborhood of* x_0.

4.2 Determinacy in the one dimensional case

In this section we are going to exploit the theory of quasi-analytic functions on $\mathbb{R}$ to prove the so-called *Carleman's Theorem*, i.e. Theorem 3.3.9 for $n = 1$. Carleman was indeed the first to approach the determinacy question with methods involving quasi-analyticity theory in his famous work of 1926 (see [6, Chapter VIII]).

Theorem 4.2.1 (Carleman's Theorem)**.**
If $\mu \in \mathcal{M}^*(\mathbb{R})$ *is such that its moment sequence* $(m_j^{\mu})_{j\in\mathbb{N}_0}$ *fulfils the following*

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{m_{2k}^{\mu}}} = \infty, \tag{4.1}$$

then μ *is* $\mathbb{R}-$*determinate.*

The original proof by Carleman makes use of the Cauchy transform of the given measure. Here, we propose a slightly different proof that uses the Fourier-Stieltjes transform but maintains the same spirit of Carleman's proof. Before proving Theorem 4.2.1, let us recall the definition of Fourier-Stieltjes transform of a measure and some fundamental properties of this object.

Definition 4.2.2. *Let $\mu \in \mathcal{M}^*(\mathbb{R})$. The Fourier-Stieltjes transform of μ is the function $F_\mu \in \mathcal{C}^\infty(\mathbb{R})$ defined by*

$$F_\mu(t) := \int_{\mathbb{R}} e^{-ixt} d\mu(x), \forall\ t \in \mathbb{R}.$$

Proposition 4.2.3. *Let $\mu, \nu \in \mathcal{M}^*(\mathbb{R})$.*
a) If $F_\mu \equiv F_\nu$ on $\mathbb{R}$, then $\mu \equiv \nu$.
b) For any $k \in \mathbb{N}_0$ and any $t \in \mathbb{R}$, we have $(D^k F_\mu)(t) = \int_{\mathbb{R}} (-ix)^k e^{-ixt} d\mu(x)$.

Proof. of Theorem 4.2.1
W.l.o.g. assume that all even moments of μ are positive. In fact, if $m^\mu_{2j} = 0$ for some $j \in \mathbb{N}_0$, then μ is supported in $\{x \in \mathbb{R} : x^{2j} = 0\} = \{0\}$ and thus, $\mu = m^\mu_0 \delta_{\{0\}}$ is the unique measure having these moments, which proves already the determinacy of μ.

Let $\nu \in \mathcal{M}^*(\mathbb{R})$ having the same moment sequence as μ and let us consider the Fourier-Stieltjes transforms of μ and ν. Then $(F_\mu - F_\nu) \in \mathcal{C}^\infty(\mathbb{R})$ and for any $k \in \mathbb{N}_0$ and any $t \in \mathbb{R}$ we get

$$(D^k(F_\mu - F_\nu))(t) = \int_{\mathbb{R}} (-ix)^k e^{-ixt} \mu(dx) - \int_{\mathbb{R}} (-ix)^k e^{-ixt} \nu(dx) \tag{4.2}$$

and so

$$\begin{aligned} \left|(D^k(F_\mu(t) - F_\nu))(t)\right| &\leq \int_{\mathbb{R}} |x|^k \mu(dx) + \int_{\mathbb{R}} |x|^k \nu(dx) \\ &\overset{\text{Hölder}}{\leq} \sqrt{m^\mu_0 m^\mu_{2k}} + \sqrt{m^\nu_0 m^\nu_{2k}} \\ &= 2\sqrt{m^\mu_0 m^\mu_{2k}} \leq (1+\gamma)\sqrt{m^\mu_{2k}}, \end{aligned}$$

where $\gamma := 2\sqrt{m^\mu_0} > 0$. Hence, $F_\mu - F_\nu \in \mathcal{C}\{s_k\}$, where $s_k := (1+\gamma)\sqrt{m^\mu_{2k}}$ for any $k \in \mathbb{N}_0$.

Since

$$\sum_{k=1}^\infty \frac{1}{\sqrt[k]{s_k}} = \sum_{k=1}^\infty \frac{1}{\sqrt[k]{(1+\gamma)\sqrt{m^\mu_{2k}}}} \geq \frac{1}{(1+\gamma)} \sum_{k=1}^\infty \frac{1}{\sqrt[2k]{m^\mu_{2k}}} \overset{(4.1)}{=} \infty,$$

Corollary 4.1.3 guarantees that the class $\mathcal{C}\{s_k\}$ is quasi-analytic.

Moreover, (4.2) gives in particular $(D^k(F_\mu - F_\nu))(0) = 0$ for all $k \in \mathbb{N}_0$. Then the quasi-analyticity of the class $C\{s_k\}$ implies that $F_\mu - F_\nu$ is identically zero on $\mathbb{R}$. Consequently, Proposition 4.2.3-a) ensures that $\mu = \nu$. □

Carleman's condition (4.1) is only sufficient for the $\mathbb{R}-$determinacy. Indeed, there exist $\mathbb{R}-$determinate measures whose moments do not fulfilll Carleman's condition (see [53] for examples).

As a consequence of Carleman's Theorem, we can derive a sufficient condition for the $(\mathbb{R}^+)-$determinacy.

Corollary 4.2.4.
Let $\mu \in \mathcal{M}^*(\mathbb{R}^+)$. If

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2k]{m_k^\mu}} = \infty, \tag{4.3}$$

then μ is $(\mathbb{R}^+)-$determinate realizing m.

Condition (4.3) is well-know as *Stieltjes' condition* since it is sufficient for the determinacy of the Stieltjes moment problem.

Before providing the proof of Corollary 4.2.4, recall that the image measure of a measure μ on $\mathcal{B}(\mathbb{R}^n)$ through a given Borel measurable map $\varphi : \mathbb{R}^n \to \mathbb{R}^d$ $(n, d \in \mathbb{N})$ is the measure $\varphi\#\mu$ on $\mathcal{B}(\mathbb{R}^d)$ defined by $\varphi\#\mu(B) := \mu(\varphi^{-1}(B))$ for all $B \in \mathcal{B}(\mathbb{R}^d)$. Moreover, for any $g : \mathbb{R}^d \to \mathbb{R}$ integrable w.r.t. $\varphi\#\mu$ we have that

$$\int_{\mathbb{R}^d} g(y) d(\varphi\#\mu)(y) = \int_{\mathbb{R}^n} (g \circ \varphi)(x) d\mu(x). \tag{4.4}$$

Proof.
Let $\mu_1, \mu_2 \in \mathcal{M}^*(\mathbb{R}^+)$ having the same moment sequence fulfilling Stieltjes' condition. For $j \in \{1, 2\}$ we define

$$d\nu_j(x) := \frac{1}{2} \left(f\#\mu_j + (-f)\#\mu_j \right),$$

where $f : \mathbb{R}^+ \to \mathbb{R}$ is given by $f(x) := \sqrt{x}$. Then (4.4) implies that for any $k \in \mathbb{N}_0$ and any $j \in \{1, 2\}$ we have

$$\begin{aligned} m_{2k}^{\nu_j} &= \int_{\mathbb{R}} y^{2k} d\nu_j(y) = \frac{1}{2} \int_{\mathbb{R}} y^{2k} d(f\#\mu_j)(y) + \frac{1}{2} \int_{\mathbb{R}} y^{2k} d((-f)\#\mu_j)(y) \\ &= \frac{1}{2} \int_{\mathbb{R}^+} (\sqrt{x})^{2k} d\mu_j(x) + \frac{1}{2} \int_{\mathbb{R}^+} (-\sqrt{x})^{2k} d\mu_j(x) = \int_{\mathbb{R}^+} (\sqrt{x})^k d\mu_j(x) = m_k^{\mu_j}. \end{aligned}$$

and

$$\begin{aligned} m_{2k+1}^{\nu_j} &= \int_{\mathbb{R}} y^{2k+1} d\nu_j(y) = \frac{1}{2} \int_{\mathbb{R}} y^{2k+1} d(f\#\mu_j)(y) + \frac{1}{2} \int_{\mathbb{R}} y^{2k+1} d((-f)\#\mu_j)(y) \\ &= \frac{1}{2} \int_{\mathbb{R}^+} (\sqrt{x})^{2k+1} d\mu_j(x) + \frac{1}{2} \int_{\mathbb{R}^+} (-\sqrt{x})^{2k+1} d\mu_j(x) = 0. \end{aligned}$$

Then ν_1 and ν_2 have the same moments and

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2k]{m_{2k}^{\nu_j}}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[2k]{m_{k}^{\mu_j}}} = \infty.$$

Hence, Carleman's Theorem 4.2.1 ensures that $\nu_1 \equiv \nu_2$ on $\mathbb{R}$ and so $\mu_1 \equiv \mu_2$ on $\mathbb{R}^+$. □

Determinacy is also deeply connected to polynomial approximation. One result in this direction is the following, which will be particularly useful in the next section.

Lemma 4.2.5.
If $\mu \in \mathcal{M}^*(\mathbb{R})$ is $\mathbb{R}-$determinate, then $\mathbb{C}[x]$ is dense in $L^2(\mathbb{R}, \mu)$.

Proof. (see e.g. [50, Proposition 6.10]) □

4.3 Determinacy in higher dimensions

In this section we are going to prove a multivariate version of Carleman's Theorem 4.2.1, namely we give a proof of Theorem 3.3.9 which we restate here for the convenience of the reader.

Theorem 4.3.1. *Let $n \in \mathbb{N}$. If $\mu \in \mathcal{M}^*(\mathbb{R}^n)$ is s.t. its moment sequence $(m_\alpha^\mu)_{\alpha \in \mathbb{N}_0^n}$ fulfills*

$$\sum_{k=1}^{\infty} m^{\mu}_{(0,\dots 0, \underbrace{2k}_{j\text{-}th}, 0,\dots,0)}{}^{-\frac{1}{2k}} = \infty, \quad \forall j \in \{1, \dots, n\}, \tag{4.5}$$

then μ is $(\mathbb{R}^n)-$determinate, i.e. the set

$$\mathcal{M}_\mu := \left\{ \nu \in \mathcal{M}^*(\mathbb{R}^n) : \int x^\alpha d\nu(x) = \int x^\alpha d\mu(x), \ \forall \alpha \in \mathbb{N}_0^n \right\}$$

is a singleton.

Note that the set $\mathcal{M}_\mu$ is convex and we have the following characterization of its extreme points[1].

[1]Recall that ν is an extreme point of $\mathcal{M}_\mu$ if the following implication holds: $(\nu = \lambda\eta_1 + (1 - \lambda\eta_2)$, for some $\lambda \in [0, 1], \eta_1, \eta_2 \in \mathcal{M}_\mu) \Rightarrow (\nu = \eta_1$ or $\nu = \eta_2)$.

Lemma 4.3.2. *Let $\mu, \nu \in \mathcal{M}^*(\mathbb{R}^n)$. Then ν is an extreme point of $\mathcal{M}_\mu$ if and only if $\mathcal{C}[X_1, \ldots, X_n]$ is dense in $L^1(\mathbb{R}^n, \nu)$.*

Proof. (see e.g. [50, Proposition 1.21]) □

To prove Theorem 4.3.1, we can proceed in the two following ways:

- We generalize the theory of quasi-analytic functions to the higher dimensions and prove an analogue of the Denjoy-Carleman theorem in the multivariate case. Using such results, we adapt the proof of Carleman's Theorem 4.2.1 to the higher dimensional case and provide a proof of Theorem 4.3.1 (see [26]).
- Using the connection between determinacy and polynomial approximation, we prove the so-called Petersen's theorem [39] about partial determinacy and so to reduce the $(\mathbb{R}^n)-$determinacy question to several $\mathbb{R}-$determinacy questions. Combining this result with Carleman's Theorem 4.2.1, we show that Theorem 4.3.1 holds (see [41]).

As we have already seen the power of the theory of quasi-analytic functions in the study of the determinacy question in the one-dimensional case, we are going now to use the second approach for the higher dimensional case. Therefore, let us first show Petersen's theorem.

Theorem 4.3.3 (Petersen's Theorem)**.**
Let $\mu \in \mathcal{M}^*(\mathbb{R}^n)$ and for each $j \in \{1, \ldots, n\}$ define $\pi_j(x) := x_j$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. If $\pi_1\#\mu, \ldots, \pi_n\#\mu$ are all $\mathbb{R}-$determinate, then μ is $(\mathbb{R}^n)-$determinate.

Proof.
Let $\nu \in \mathcal{M}_\mu$ and $j \in \{1, \ldots, n\}$. Then for any $k \in \mathbb{N}_0$ we have that

$$\begin{aligned}
\int_{\mathbb{R}} y^k d(\pi_j\#\nu)(y) &= \int_{\mathbb{R}^n} \pi_j(x)^k d\nu(x) \\
&= \int_{\mathbb{R}^n} x^{(0,\ldots,0,k,0,\ldots,0)} d\nu(x) \\
&= \int_{\mathbb{R}^n} x^{(0,\ldots,0,k,0,\ldots,0)} d\mu(x) \\
&= \int_{\mathbb{R}^n} \pi_j(x)^k d\mu(x) \\
&= \int_{\mathbb{R}} y^k d(\pi_j\#\mu)(y),
\end{aligned}$$

i.e. $(\pi_j\#\nu) \in \mathcal{M}_{\pi_j\#\mu}$. This implies that

$$\pi_j\#\nu = \pi_j\#\mu \tag{4.6}$$

as $\pi_j\#\mu$ is $\mathbb{R}-$determinate. Moreover, the determinacy of $\pi_j\#\mu$ implies that $\mathbb{C}[X_j]$ is dense in $L^2(\mathbb{R},\mu)$ by Lemma 4.2.5 and so that

$$\forall \varepsilon > 0, \forall B_j \in \mathcal{B}(\mathbb{R}), \exists\, p_j \in \mathbb{C}[X_j] \text{ s.t. } \left\| \mathbb{1}_{B_j} - p_j \right\|_{L^2(\mathbb{R},\pi_j\#\mu)} \leq \varepsilon. \tag{4.7}$$

Since

$$\begin{aligned}
\left\| \mathbb{1}_{B_j} - p_j \right\|_{L^2(\mathbb{R},\pi_j\#\mu)} &\overset{(4.6)}{=} \left\| \mathbb{1}_{B_j} - p_j \right\|_{L^2(\mathbb{R},\pi_j\#\nu)} \\
&= \left(\int_{\mathbb{R}} (\mathbb{1}_{B_j}(y) - p_j(y))^2 d(\pi_j\#\nu)(y) \right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{R}^n} (\mathbb{1}_{B_j}(\pi_j(x)) - p_j(\pi_j(x)))^2 d\nu(x) \right)^{\frac{1}{2}} \\
&= \left\| \mathbb{1}_{B_j} \circ \pi_j - p_j \circ \pi_j \right\|_{L^2(\mathbb{R}^n,\nu)},
\end{aligned}$$

we can rewrite (4.7) as

$$\forall \varepsilon > 0, \forall B_j \in \mathcal{B}(\mathbb{R}), \exists\, p_j \in \mathbb{C}[X_j] \text{ s.t. } \left\| \mathbb{1}_{B_j} \circ \pi_j - p_j \circ \pi_j \right\|_{L^2(\mathbb{R}^n,\nu)} \leq \varepsilon. \tag{4.8}$$

Now the function $(\mathbb{1}_{B_1} \circ \pi_1) \cdots (\mathbb{1}_{B_n} \circ \pi_n) - (p_1 \circ \pi_1) \cdots (p_n \circ \pi_n)$ on $\mathbb{R}^n$ can be rewritten as

$$\begin{aligned}
&(\mathbb{1}_{B_1} \circ \pi_1) \cdots (\mathbb{1}_{B_n} \circ \pi_n) - (p_1 \circ \pi_1) \cdots (p_n \circ \pi_n) = \\
&(\mathbb{1}_{B_1} \circ \pi_1 - p_1 \circ \pi_1)(\mathbb{1}_{B_2} \circ \pi_2) \cdots (\mathbb{1}_{B_n} \circ \pi_n) + \\
&\qquad + \quad (p_1 \circ \pi_1)(\mathbb{1}_{B_2} \circ \pi_2 - p_2 \circ \pi_2)(\mathbb{1}_{B_3} \circ \pi_3) \cdots (\mathbb{1}_{B_n} \circ \pi_n) + \\
&\qquad + \cdots + \quad (p_1 \circ \pi_1) \cdots (p_{n-1} \circ \pi_{n-1})(\mathbb{1}_{B_n} \circ \pi_n - p_n \circ \pi_n).
\end{aligned} \tag{4.9}$$

and so

$$\begin{aligned}
&\|(\mathbb{1}_{B_1} \circ \pi_1) \cdots (\mathbb{1}_{B_n} \circ \pi_n) - (p_1 \circ \pi_1) \cdots (p_n \circ \pi_n)\|_{L^1(\mathbb{R}^n,\nu)} \\
\overset{(4.9)}{\leq} \quad & \|(\mathbb{1}_{B_1} \circ \pi_1 - p_1 \circ \pi_1)(\mathbb{1}_{B_2} \circ \pi_2) \cdots (\mathbb{1}_{B_n} \circ \pi_n)\|_{L^1(\mathbb{R}^n,\nu)} + \cdots \\
\cdots \quad & + \|(p_1 \circ \pi_1) \cdots (p_{n-1} \circ \pi_{n-1})(\mathbb{1}_{B_n} \circ \pi_n - p_n \circ \pi_n)\|_{L^1(\mathbb{R}^n,\nu)} \\
\overset{\text{Hölder}}{\leq} \quad & \|\mathbb{1}_{B_1} \circ \pi_1 - p_1 \circ \pi_1\|_{L^2(\mathbb{R}^n,\nu)} \|(\mathbb{1}_{B_2} \circ \pi_2) \cdots (\mathbb{1}_{B_n} \circ \pi_n)\|_{L^2(\mathbb{R}^n,\nu)} + \cdots \\
\cdots \quad & + \|(p_1 \circ \pi_1) \cdots (p_{n-1} \circ \pi_{n-1})\|_{L^2(\mathbb{R}^n,\nu)} \|\mathbb{1}_{B_n} \circ \pi_n - p_n \circ \pi_n\|_{L^2(\mathbb{R}^n,\nu)} \\
\overset{(4.8)}{\leq} \quad & C\varepsilon,
\end{aligned}$$

where $C > 0$.

This shows that $\mathbb{C}[X_1, \ldots, X_n]$ is dense in the subset

$$\mathcal{S} := \{(\mathbb{1}_{B_1} \circ \pi_1) \cdots (\mathbb{1}_{B_n} \circ \pi_n) : B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}^n)\}$$

of $L^1(\mathbb{R}^n, \nu)$. Since $\operatorname{span}(\mathcal{S})$ is dense in $L^1(\mathbb{R}^n, \nu)$, we get that $\mathbb{C}[X_1, \ldots, X_n]$ is dense in $L^1(\mathbb{R}^n, \nu)$ and so by Lemma 4.3.2 we obtain that ν is an extreme point of $\mathcal{M}_\mu$.

Since ν was arbitrary in $\mathcal{M}_\mu$, we have showed that every point of $\mathcal{M}_\mu$ is extreme. In particular, $\eta := \frac{1}{2}(\mu + \nu) \in \mathcal{M}_\mu$ is extreme and so $\eta = \mu$ or $\eta = \nu$, which imply $\nu = \mu$. Hence, μ is $(\mathbb{R}^n)-$determinate. □

Proof. of Theorem 4.3.1
For any $j \in \{1, \ldots, n\}$ and for any $k \in \mathbb{N}$ we have that

$$\begin{aligned} m_{2k}^{\pi_j \# \mu} &= \int_{\mathbb{R}} y^{2k} d(\pi_j \# \mu)(y) = \int_{\mathbb{R}^n} (\pi_j(x))^{2k} d\mu(x) \\ &= \int_{\mathbb{R}^n} x^{(0,\ldots,0,2k,0,\ldots,0)} d\mu(x) = m^{\mu}_{(0,\ldots,0,2k,0,\ldots,0)}. \end{aligned}$$

Hence, the assumption that μ fulfils (4.5) gives that each $\pi_j \# \mu$ fulfils (4.1). Therefore, Carleman's Theorem 4.2.1 guarantees that each $\pi_j \# \mu$ is $\mathbb{R}-$determinate and so by Petersen's Theorem 4.3.3 we obtain that μ is $(\mathbb{R}^n)-$determinate. □