1.3 Relation between $\mathrm{Psd}(K_S)$ and T_S

Fixed a finite subset $S := \{g_1, \ldots, g_s\} \subset \mathbb{R}[X_1, \ldots, X_n]$, we want to study the relation between the (quadratic) preordering associated to S, i.e.

$$T_S := \left\{ \sum_{e=(e_1,\ldots,e_s)\in\{0,1\}^s} \sigma_e \, g_1^{e_1} \ldots g_s^{e_s} : \sigma_e \in \sum \mathbb{R}[\underline{X}]^2, \forall e \in \{0,1\}^s \right\},$$

and $\mathrm{Psd}(K_S)$ where

$$K_S := \{x \in \mathbb{R}^n : g_i(x) \geq 0, \, i = 1, \ldots, s\}.$$

The first result in this direction is the so-called *Stengle Positivstellensatz*, whose proof is due to Stengle in 1974 even if most ideas were already contained in an article of Krivine of 1964.

Theorem 1.3.1. *Let $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[\underline{X}]$ and $f \in \mathbb{R}[\underline{X}]$. Then:*
(1) $f > 0$ on $K_S \Leftrightarrow \exists \; p, q \in T_S$ s.t. $pf = 1 + q$ (Striktpositivstellensatz)
(2) $f \geq 0$ on $K_S \Leftrightarrow \exists \; m \in \mathbb{N}_0, \exists \; p, q \in T_S$ s.t. $pf = f^{2m} + q$ (Nonnegativstellensatz)
(3) $f = 0$ on $K_S \Leftrightarrow \exists \; m \in \mathbb{N}_0$ s.t. $-f^{2m} \in T_S$ (Real Nullstellensatz)
(4) $K_S = \phi \Leftrightarrow -1 \in T_S$.

Taking $S = \emptyset$ in (2) we obtain an alternative proof for Artin's solution (1927) to the Hilbert's 17th problem posed in 1900 of establishing whether or not a psd polynomial is always a sum of squares of rational functions.

Corollary 1.3.2. *Let $f \in \mathbb{R}[\underline{X}]$.*
If $f(x) \geq 0$ for all $x \in \mathbb{R}^n$ then $f \in \sum \mathbb{R}(\underline{X})^2$.

Proof. Suppose that $f(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $f \not\equiv 0$. By taking $S = \emptyset$ in (2), we get that $\exists \; m \in \mathbb{N}_0, \exists \; p, q \in T_S = \sum \mathbb{R}[\underline{X}]^2$ s.t. $pf = f^{2m} + q$. Since $f \not\equiv 0$, also $f^{2m} + q \not\equiv 0$ and $p \not\equiv 0$. Hence,

$$f = \frac{f^{2m} + q}{p} = \left(\frac{1}{p}\right)^2 p(f^{2m} + q) \in \sum \mathbb{R}(\underline{X})^2.$$

If $f \equiv 0$ then clearly the conclusion holds. □

Theorem 1.3.1-(2) gives a representation of elements in $\mathrm{Psd}(K_S)$ as quotients of elements in T_S. Therefore, it is natural to look for denominator free Positivstellensätze. In particular, for the rest of this section we are going to focus on saturation of preorderings.

Definition 1.3.3. *Let $S \subset \mathbb{R}[\underline{X}]$ be finite. The preordering T_S in $\mathbb{R}[\underline{X}]$ is said to be* saturated *if* $\mathrm{Psd}(K_S) = T_S$.

In [5, Lecture 24, 25] the following result was proved in details:

Proposition 1.3.4. *Suppose $n \geq 3$. Let S be a finite subset of $\mathbb{R}[\underline{X}]$ such that $K_S \subseteq \mathbb{R}^n$ and $\mathrm{int}(K_S) \neq \emptyset$. Then there exists $f \in \mathbb{R}[\underline{X}]$ such that $f \geq 0$ on $\mathbb{R}^n$ and $f \notin T_S$.*

This already excludes saturation already for an entire class of preorderings and can be actually obtained as a corollary of the following more general result due to Scheiderer [11, Proposition 6.1].

Theorem 1.3.5. *Let S be a finite subset of $\mathbb{R}[\underline{X}]$ s.t. $\dim(K_S) \geq 3$. Then there exists $f \in \mathbb{R}[\underline{X}]$ s.t. $f \geq 0$ on $\mathbb{R}^n$ and $f \notin T_S$.*

Recall that the dimension of a bcsas $K \subseteq \mathbb{R}^n$ is defined as the Krull dimension of $\frac{\mathbb{R}[\underline{X}]}{\mathcal{I}(K)}$ where $\mathcal{I}(K)$ is the ideal of polynomials vanishing on K. To derive Proposition 1.3.4 from Theorem 1.3.5, it is enough to prove that $\mathrm{int}(K_S) \neq \emptyset$ implies $\dim(K_S) = 3$ (see [5, Lemma 2.7]).

For lower dimensional bcsas, there are examples in which saturation holds and examples in which it fails. An example of one dimensional bcsas which can be described both by a saturated preordering and by a non-saturated preordering is $\mathbb{R}^+$.

Example 1.3.6. *Let $K = [0, +\infty)$. For $S_1 := \{X\}$, we have that $K = K_{S_1}$ and Proposition 1.2.4-a) ensures that $\mathrm{Psd}([0, +\infty)) = T_{S_1}$. Hence, T_{S_1} is saturated. However, by taking the representation $K = K_{S_2}$ with $S_2 := \{X^3\}$, we do not have anymore the saturation of the corresponding preordering. In fact, $X \in \mathrm{Psd}(K)$ but $X \notin T_{S_2}$.*

Suppose that there exist $\sigma_1, \sigma_2 \in \sum \mathbb{R}[X]^2$ s.t. $X = \underbrace{\sigma_1 + X^3\sigma_2}_{=:q}$. Then we have four possibilities:

- *if $\sigma_1 \equiv 0 \equiv \sigma_2$ then $q(X) \equiv 0$.*
- *if $\sigma_1 \equiv 0$ and $\sigma_2 \not\equiv 0$ then $\deg(q)$ is odd and ≥ 3.*
- *if $\sigma_1 \not\equiv 0$ and $\sigma_2 \equiv 0$ then $\deg(q)$ is even.*
- *if $\sigma_1 \not\equiv 0$ and $\sigma_2 \not\equiv 0$ then $\deg(q) = \max\{\deg(\sigma_1), \deg(X^3\sigma_2)\}$ which is either even or odd ≥ 3.*

Hence, $X \not\equiv q$) which leads to the desired contradiction.

In the one variable case, it is possible to show that for any bcsas K of $\mathbb{R}[X]$ there exists $S \subset \mathbb{R}[X]$ finite such that $K = K_S$ and T_S is saturated. Such a S is called the *natural description* of K.

Definition 1.3.7. *Let K be a non-empty bcsas of $\mathbb{R}$, i.e. K is a finite union of intervals and points. The* natural description *of K is defined as the finite subset S_{nat} of $\mathbb{R}[X]$ s.t.*

(i) if $a \in \mathbb{R}$ is the smallest element of K, then $X - a \in S_{nat}$
(ii) if $a \in \mathbb{R}$ is the greatest element of K, then $a - X \in S_{nat}$
(iii) if $a, b \in K$, $a < b$ and $(a, b) \cap K = \phi$, then $(X - a)(X - b) \in S_{nat}$
(iv) no other polynomial is in S_{nat}.

Examples 1.3.8.

- *If $K = [0, +\infty)$ then $S_{nat} = \{X\}$, since 0 is the smallest element of K, K has no greatest element and for all $a, b \in K$ with $a < b$ we have $(a, b) \cap K \neq \emptyset$.*
- *If $K = [0, 1]$ then $S_{nat} = \{X, 1 - X\}$, since 0 is the smallest element of K, 1 is the greatest element of K and for all $a, b \in K$ with $a < b$ we have $(a, b) \cap K \neq \emptyset$.*
- *If $K = -1 \cup [0, 1]$ then $S_{nat} = \{X + 1, 1 - X, X(X + 1)\}$, since -1 is the smallest element of K, 1 is the greatest element of K and $(-1, 0) \cap K = \emptyset$.*

Theorem 1.3.9. *Let K be a non-empty bcsas of $\mathbb{R}$. Then the preordering associated to the natural description S_{nat} of K is saturated.*

Proof. For notational convenience, set S equal to the natural description S_{nat} of K. We want to show that $\mathrm{Psd}(K) = T_S$.

If $K = \mathbb{R}$ then $S = \emptyset$ and $T_S = \sum \mathbb{R}[X]^2$, so the conclusion holds. Therefore, we can assume that $K \subsetneq \mathbb{R}$. Then Definition 1.3.7 provides the following information:

- If K has a smallest element a, then $X - a \in S$ and so
$$\forall d \leq a, X - d = (X - a) \cdot 1^2 + (a - d) \in T_S. \tag{1.2}$$
- if K has a greatest element a, then $a - X \in S$ and so
$$\forall d \geq a, d - X = (a - X) \cdot 1^2 + (d - a) \in T_S. \tag{1.3}$$
- if $a, b \in K$, $a < b$ and $(a, b) \cap K = \phi$, then $(X - a)(X - b) \in S$ and, by Exercise 1 in Sheet 1 we have that
$$\forall d, e \in \mathbb{R} \text{ s.t. } a \leq d \leq e \leq b, (X - d)(X - e) \in T_S. \tag{1.4}$$

Suppose that $f \in \mathrm{Psd}(K)$ and proceed by induction on $\deg(f)$.

If $\deg(f) = 0$ then $f(x) = k$ for all $x \in \mathbb{R}^d$ with $k \geq 0$. Hence, $f \in \sum \mathbb{R}[X]^2 \subset T_S$.

Suppose that $\deg(f) = m \geq 1$ and that for all $g \in \text{Psd}(K)$ with $\deg(g) \leq m-1$ we know that $g \in T_S$. W.l.o.g. we can assume that there exists $c \in \mathbb{R}$ s.t. $f(c) < 0$ (otherwise $f \geq 0$ on $\mathbb{R}$ which gives $f \in \sum \mathbb{R}[X]^2 \subset T_S$). Then there are the following three possibilities: either K has a least element a and $c < a$ or K has a largest element a and $c > a$ or there exist $a, b \in K$ with $a < b$, $(a, b) \cap K = \emptyset$ and $a < c < b$.

Case 1: if K has a least element a and $c < a$, then f has a root d in the interval $(c, a]$. Therefore, $f = (X-d)g$ for some $g \in \mathbb{R}[X]$ with $\deg(g) = m-1$. As $f \geq 0$ on K and $X - d \geq 0$ on K, we get that $g \in \text{Psd}(K)$. Hence, by inductive assumption we have that $g \in T_S$. Also, $X - d \in T_S$ by (1.2) and so $f \in T_S$.

Case 2: If K has a largest element a and $c > a$, then f has a root d in the interval $[a, c)$. Therefore, $f = (d-X)g$ for some $g \in \mathbb{R}[X]$ with $\deg(g) = m-1$. As $f \geq 0$ on K and $d - X \geq 0$ on K, we get that $g \in \text{Psd}(K)$. Hence, by inductive assumption we have that $g \in T_S$. Also, $d - X \in T_S$ by (1.3) and so $f \in T_S$.

Case 3: If there exist $a, b \in K$ with $a < b$, $(a, b) \cap K = \emptyset$ and $a < c < b$, then f has a greatest root d in the interval $[a, c)$ and a least root e in the interval $(c, b]$. Therefore, $f = (X - d)(X - e)g$ for some $g \in \mathbb{R}[X]$ with $\deg(g) = m - 2$. As $f \geq 0$ on K and $(X - d)(X - e) \geq 0$ on K, we get that $g \in \text{Psd}(K)$. Hence, by inductive assumption we have that $g \in T_S$. Also, $(X - d)(X - e) \in T_S$ by (1.4) and so $f \in T_S$. □

Corollary 1.3.10. *Let K be a non-empty bcsas of $\mathbb{R}$. If $S \subset \mathbb{R}[X]$ is finite s.t. $K = K_S$ and $S \supseteq S_{nat}$ (up to a positive scalar multiple factor), then T_S is saturated.*

Proof. By Theorem 1.3.9, we know that $\text{Psd}(K) = T_{S_{nat}}$. As $S \supseteq S_{nat}$ (up to a positive scalar multiple factor), we also have that $T_{S_{nat}} \subseteq T_S$. Hence, $\text{Psd}(K) = T_S$, i.e. T_S is saturated. □

Note that the converse of this result does not hold in general. In fact, if S does not contain the natural description then T_S might be or not be saturated as showed by the following example. However, for non-compact bcsas of $\mathbb{R}$ the converse holds.

Example 1.3.11. *Let $K = [0, 1]$. Then $S_{nat} = \{X, 1 - X\}$ is the natural description of K. Hence, by Theorem 1.3.9, $T_{S_{nat}}$ is saturated. If we take now $S_1 := \{X^3, 1 - X\}$, then $K = K_{S_1}$, S_1 does not contain S_{nat} and T_{S_1} is not saturated (see Sheet 1, Exercise 2 for a proof). However, also $S_2 = \{X(1-X)\}$*

does not contain S_{nat} *and* $K = K_{S_2}$*, but* T_{S_2} *is saturated. Indeed, we have that* $X = X^2 + X(1-X) \in T_{S_2}$ *and* $1 - X = (1-X)^2 + X(1-X) \in T_{S_2}$*, which imply* $T_{S_{nat}} \subseteq T_{S_2}$ *and so that* $\text{Psd}(K) = T_{S_2}$*.*

Proposition 1.3.12. *Let* $K \subseteq \mathbb{R}$ *be a non-compact bcsas of* $\mathbb{R}[X]$ *and* $S \subset \mathbb{R}[X]$ *finite s.t.* $K = K_S$*. Then* T_S *is saturated* $\Leftrightarrow S \supseteq S_{nat}$ *(up to a positive scalar multiple factor).*

Proof. One direction always holds by Corollary 1.3.10, while for the converse the non-compactness is essential.

Suppose that K_S is not compact and $\text{Psd}(K_S) = T_S$. We can assume that for any $g \in S$ we have $\deg(g) \geq 1$. Since K_S is not compact, it either contains an interval of the form $[c, +\infty)$ or it contains an interval of the form $(-\infty, c]$. Replacing X by $-X$ when necessary in the following proof, we can assume that we are in the first case. This implies that every $g \in S$ is non-negative on $[c, +\infty)$ and so has positive leading coefficient.

Suppose that K_S has a smallest element a and consider $p := X - a$. Then $p \in \text{Psd}(K_S)$ and so by assumption we have $p \in T_S$. This together with the fact that $\deg(p) = 1$ and that $\deg(g) \geq 1$, for all $g \in S$ ensures that $p = \sigma_1 g_1 + \ldots + \sigma_t g_t$, where $\sigma_1, \ldots, \sigma_t \in \mathbb{R}^+$ and $g_i \in S$ with $\deg(g_i) = 1$ for $i = 1, \ldots, t$. As $p(a) = 0$ and $g_i(a) \geq 0$ for all $i = 1, \ldots, t$ (since $a \in K_S$), we can conclude that there exists at least one $i \in \{1, \ldots, t\}$ such that $g_i(a) = 0$. Hence, there exists $r > 0$ such that $g_i = r(X - a)$, i.e. $r(X - a) \in S$ as required.

... TO BE CONTINUED IN THE NEXT LECTURE. □

Applying the so-called Scheiderer's Local Global Principle (see e.g. [7, Section 9]), one can provide examples of two dimensional compact bcsas which can be described by a saturated preordering.

Examples 1.3.13.

1. *The preordering* T_S *for* $S = \{X, 1-X, Y, 1-Y\}$ *is saturated. Here* K_S *is the unit square in* $\mathbb{R}^2$*.*
2. *The preordering* T_S *for* $S = \{1 - X^2 - Y^2\}$ *is saturated. Here* K_S *is the unit disk in* $\mathbb{R}^2$*.*

However, there are examples of two dimensional compact bcsas for which saturation does not hold.

Example 1.3.14. *Let* $S := \{X^3 - Y^3, 1 - X\}$. *Then* K_S *is compact in* $\mathbb{R}^2$ *and* T_S *is not saturated. Indeed, the polynomial* $X \in \mathbb{R}[X,Y]$ *is nonnegative on* K_S *but does not belong to* T_S.

Suppose that there exists $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \sum \mathbb{R}[X,Y]^2$ *s.t.*

$$X = \underbrace{\sigma_1 + (X^3 - Y^3)\sigma_2 + (1 - X)\sigma_3 + (X^3 - Y^3)(1 - X)\sigma_4}_{=:q}.$$

Evaluating at $Y = 0$, *we have that* $X \equiv q(X,0) = \sigma_1(X,0) + X^3\sigma_2(X,0) + (1 - X)\sigma_3(X,0) + X^3(1 - X)\sigma_4(X,0)$, *i.e.* X *belongs to the preordering generated by* $\{X^3, 1 - X\}$ *in* $\mathbb{R}[X]$ *which is false as showed in Sheet 1, Exercise 2.*

For non-compact two dimensional bcsas, we have both saturated and non-saturated associated preorderings.

Examples 1.3.15.

1. *If* $S = \emptyset \subset \mathbb{R}[X,Y]$ *then* $T_S = \sum \mathbb{R}[X,Y]^2$ *is not saturated as* $K_S = \mathbb{R}^2$.
2. *If* $S = \{X(1 - X)\} \subset \mathbb{R}[X,Y]$, *then* $\mathrm{Psd}(\underbrace{[0,1] \times \mathbb{R}}_{=K_S}) = T_S$, *i.e.* T_S *is saturated (see [8]).*

Summarizing we have that a preordering T_S in $\mathbb{R}[\underline{X}]$ is always not saturated if $\dim(K_S) \geq 3$, but can be or cannot be saturated if $\dim(K_S) \in \{1, 2\}$ (depending on the geometry of K_S and the chosen description S).