does not contain S_{nat} and $K = K_{S_2}$, but T_{S_2} is saturated. Indeed, we have that $X = X^2 + X(1-X) \in T_{S_2}$ and $1 - X = (1-X)^2 + X(1-X) \in T_{S_2}$, which imply $T_{S_{nat}} \subseteq T_{S_2}$ and so that $\mathrm{Psd}(K) = T_{S_2}$.

Proposition 1.3.12. *Let $K \subseteq \mathbb{R}$ be a non-compact bcsas of $\mathbb{R}[X]$ and S a finite subset of $\mathbb{R}[X]$ s.t. $K = K_S$. Then T_S is saturated $\Leftrightarrow S \supseteq S_{nat}$ (up to a positive scalar multiple factor).*

Before proving this result, let us introduce the notion of width of a quadratic polynomial in one variable and an elementary related property which will be useful in the proof of Proposition 1.3.12.

Definition 1.3.13. *Let $f \in \mathbb{R}[X]$ be such that $\deg(f) = 2$. If r_1, r_2 are the real roots of f and $r_1 \leq r_2$, then* width of f *is denoted by $w(f)$ and defined to be $r_2 - r_1$. If f has no real roots, then $w(f) := 0$.*

Lemma 1.3.14. *Let $f_1, f_2 \in \mathbb{R}[X]$ with $\deg(f_1) = 2 = \deg(f_2)$ and positive leading coefficients. Then $w(f_1 + f_2) \leq \max\{w(f_1), w(f_2)\}$.*

Proof. W.l.o.g. we can assume that $w(f_1) \geq w(f_2)$ and that $w(f_1) > 0$ (otherwise $w(f_1) = w(f_2) = 0$ and so $f_1 + f_2$ has either one root or no roots, i.e. $w(f_1 + f_2) = 0$). Shifting and scaling we can always reduce to the case $f_1 := X^2 - X$ and $f_2 := c(X-a)(X-(a+b))$ with $a, b, c \in \mathbb{R}$ such that $0 \leq b \leq 1$ and $c > 0$. Thus, we get

$$f_1 + f_2 = (c+1)X^2 - (2ac + bc + 1)X + ca(a+b),$$

whose roots are $\frac{2ac+bc+1 \pm \sqrt{(2ac+bc+1)^2 - 4ca(a+b)(c+1)}}{2(c+1)}$ and so

$$w(f_1 + f_2) = \frac{\sqrt{(2ac+bc+1)^2 - 4ca(a+b)(c+1)}}{(c+1)}.$$

We want to show that $w(f_1 + f_2) \leq w(f_1) = 1$, which by expanding is equivalent to show $(1-b^2)(c+1) + (2a+b-1)^2 \geq 0$. The latter indeed holds since $c > 0$ and $0 \leq b \leq 1$. $\square$

Proof. of Proposition 1.3.12.
One direction always holds by Corollary 1.3.10, while for the converse the non-compactness is essential.

Suppose that K_S is not compact and $\mathrm{Psd}(K_S) = T_S$. We can assume that for any $g \in S$ we have $\deg(g) \geq 1$. Since K_S is not compact, it either contains an interval of the form $[c, +\infty)$ or it contains an interval of the form $(-\infty, c]$. Replacing X by $-X$ when necessary in the following proof, we can assume that we are in the first case. This implies that every $g \in S$ is non-negative on $[c, +\infty)$ and so has positive leading coefficient.

Suppose that K_S has a smallest element a and consider $p := X - a$. Then $p \in \mathrm{Psd}(K_S)$ and so by assumption we have $p \in T_S$. This together with the fact that $\deg(p) = 1$ and that $\deg(g) \geq 1$, for all $g \in S$ ensures that $p = \sigma_1 g_1 + \ldots + \sigma_t g_t$, where $\sigma_1, \ldots, \sigma_t \in \mathbb{R}^+$ and $g_i \in S$ with $\deg(g_i) = 1$ for $i = 1, \ldots, t$. As $p(a) = 0$ and $g_i(a) \geq 0$ for all $i = 1, \ldots, t$ (since $a \in K_S$), we can conclude that there exists at least one $i \in \{1, \ldots, t\}$ such that $g_i(a) = 0$. Hence, there exists $r > 0$ such that $g_i = r(X - a)$, i.e. $r(X - a) \in S$ as required.

Suppose now that $a, b \in K_S$ are such that $a < b$ and $(a, b) \cap K_S = \emptyset$ and set $p := (X - a)(X - b)$. Then $p \in \mathrm{Psd}(K_S)$ and so by assumption $p \in T_S$. This together with the fact that $\deg(p) = 2$ and that $\deg(g) \geq 1, \forall g \in S$ ensures that p is a sum of terms of the form σf and $\xi g h$ with $\sigma, \xi \in \mathbb{R}^+$ and $f, g, h \in S$ with $\deg(f) \in \{1, 2\}$ and $\deg(g) = 1 = \deg(h)$. Since any linear $g \in S$ is increasing and $g(a) \geq 0$, g is positive on the interval (a, b). Thus, $p \geq \sigma_1 g_1 + \cdots + \sigma_t g_t$ on (a, b), where $\sigma_1, \ldots, \sigma_t \in \mathbb{R}^+ \setminus \{0\}$ and $g_1, \ldots, g_t \in S$ are quadratics which assume at least one negative value on (a, b). Now for each $i \in \{1, \ldots, t\}$, we have that g_i opens upward, $g_i(a) \geq 0$ and $g_i(b) \geq 0$, which imply that g_i has its roots in $[a, b]$ and consequently $w(g_i) \leq b - a$ (see Definition 1.3.13). Since $w(p) = b - a$ and $p \geq \sigma_1 g_1 + \cdots + \sigma_t g_t$ on (a, b), we have that necessarily $w(\sigma_1 g_1 + \cdots + \sigma_t g_t) = b - a$. Hence, by Lemma 1.3.14, we get

$$b - a = w(\sigma_1 g_1 + \cdots + \sigma_t g_t) \leq \max_{i=1,\ldots,t} w(\sigma_i g_i) = \max_{i=1,\ldots,t} w(g_i) \leq b - a,$$

which implies that there exists $i \in \{1, \ldots, t\}$ such that $w(g_i) = b - a$. Hence, g_i necessarily has the form $r(X - a)(X - b)$ for some real $r > 0$, that is, $r(X - a)(X - b) \in S$ as required. $\square$

Applying the so-called Scheiderer's Local Global Principle (see e.g. [29, Section 9]), one can provide examples of two dimensional compact bcsas which can be described by a saturated preordering.

Examples 1.3.15.

1. *The preordering T_S for $S = \{X, 1 - X, Y, 1 - Y\}$ is saturated. Here K_S is the unit square in $\mathbb{R}^2$.*
2. *The preordering T_S for $S = \{1 - X^2 - Y^2\}$ is saturated. Here K_S is the unit disk in $\mathbb{R}^2$.*

However, there are examples of two dimensional compact bcsas for which saturation does not hold.

Example 1.3.16. *Let $S := \{X^3 - Y^3, 1 - X\}$. Then K_S is compact in $\mathbb{R}^2$ and T_S is not saturated. Indeed, the polynomial $X \in \mathbb{R}[X,Y]$ is nonnegative on K_S but does not belong to T_S.*

Suppose that there exists $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \sum \mathbb{R}[X,Y]^2$ s.t.

$$X = \underbrace{\sigma_1 + (X^3 - Y^3)\sigma_2 + (1 - X)\sigma_3 + (X^3 - Y^3)(1 - X)\sigma_4}_{=:q}.$$

Evaluating at $Y = 0$, we have that $X \equiv q(X,0) = \sigma_1(X,0) + X^3\sigma_2(X,0) + (1 - X)\sigma_3(X,0) + X^3(1 - X)\sigma_4(X,0)$, i.e. X belongs to the preordering generated by $\{X^3, 1 - X\}$ in $\mathbb{R}[X]$ which is false as showed in Sheet 1, Exercise 2.

For non-compact two dimensional bcsas, we have both saturated and non-saturated associated preorderings.

Examples 1.3.17.

1. *If $S = \emptyset \subset \mathbb{R}[X,Y]$ then $T_S = \sum \mathbb{R}[X,Y]^2$ is not saturated as $K_S = \mathbb{R}^2$.*
2. *If $S = \{X(1 - X)\} \subset \mathbb{R}[X,Y]$, then $\mathrm{Psd}(\underbrace{[0,1] \times \mathbb{R}}_{=K_S}) = T_S$,*

   i.e. T_S is saturated (see [30]).

Summarizing we have that a preordering T_S in $\mathbb{R}[\underline{X}]$ is always not saturated if $\dim(K_S) \geq 3$, but can be or cannot be saturated if $\dim(K_S) \in \{1, 2\}$ (depending on the geometry of K_S and the chosen description S).

1.3.2 Representation Theorem and Positivstellensätze

We have seen that saturation of preorderings does not occur for a large class of bcsas. Therefore, in the cases when saturation does not occur, it is still standing our question of how to characterise $\mathrm{Psd}(K_S)$ in terms of T_S without using quotients of its elements. For compact bcsas, a denominator free Positivstellensatz was provided by Schmüdgen in [36] as a corollary of a fundamental result for the $K-$MP for K compact bcsas. This rather surprising result had a great impact in this area and it can be considered a breakthrough in both the theory of positive polynomials and the moment problem. Generalizations of this result were proved by Putinar in [33] and Jacobi in [17] in the coming ten years. Moreover, the Schmüdgen Positivstellensatz gave the impulse to a lively research activity about the moment problem in the non-compact case.

In this section, we are not providing the original Schmüdgen proof but we will derive his Positivstellensatz from a general version of the so-called

Representation Theorem due to Marshall [27]. Actually, Schmüdgen's Positivstellensatz can be obtained as a corollary of a less general and earlier version of the Representation Theorem due to Krivine [21, 22]. This was first noticed by Wörmann in [40], but there was no obvious way to derive Putinar's Positivstellensatz from the Krivine Representation Theorem. Only in 2001 with Jacobi's generalized version of the Representation Theorem [17] it became possible to give a completely algebraic proof of Putinar's Positivstellensatz. The further extension of the Representation Theorem we give here (see Theorem 1.3.24) allows to derive all the above mentioned Positivstellensätze as well as a nice refinement of Putinar's result (see Theorem **??**). In order to state such a Representation Theorem we need to introduce the following general setting.

Let A be a commutative ring with 1 and for simplicity let us assume that $\mathbb{Q} \subseteq A$. We denote by $X(A)$ the *character space* of A, i.e. the set of all unitary ring homomorphisms from A to $\mathbb{R}$. For any $a \in A$, we define the *Gelfand transform* $\hat{a} : X(A) \to \mathbb{R}$ as $\hat{a}(\alpha) := \alpha(a)$, $\forall \alpha \in X(A)$.

For any subset M of A, we set

$$\mathcal{K}_M := \{\alpha \in X(A) : \hat{a}(\alpha) \geq 0,\ \forall a \in M\}.$$

If $M = \sum A^{2d}$ then $\mathcal{K}_M = X(A)$. If M is the $2d-$power module of A generated by $\{p_j\}_{j\in J}$ then $\mathcal{K}_M = \{\alpha \in X(A) : \hat{p_j}(\alpha) \geq 0,\ \forall\ j \in J\}$.

If $a \in M$, then clearly $\hat{a} \geq 0$ on $\mathcal{K}_M$. Does the converse hold, i.e. is it true that if $a \in A$ is such that $\hat{a} \geq 0$ on $\mathcal{K}_M$, then $a \in M$? The Representation Theorem exactly provides an answer to this question. In order to rigorously formulate this result, we need some further notions and properties.

Definition 1.3.18. *A* preprime *of A is a subset T of A such that $T + T \subseteq T$, $T \cdot T \subseteq T$ and $\mathbb{Q}^+ \subseteq T$.*

Definition 1.3.19. *Let T be a preprime of A.*

- *A* $T-$module *of A is a subset M of A such that $M + M \subseteq M$, $T \cdot M \subseteq M$ and $1 \in M$.*
- *A $T-$module is said to be* Archimedean *if for each $a \in A$ there exists $N \in \mathbb{N}$ such that $N \pm a \in M$.*

Remark 1.3.20.

- *A preprime T is itself a $T-$module.*
- *If a preprime T is Archimedian, then any $T-$module is also Archimedian since $T \subseteq M$.*

Note that, for $d \in \mathbb{N}$, any $2d-$power preordering of A is a preprime and any $2d-$power module of A is a $\sum_A^{2d}-$module (see Definition 1.2.5). In particular, any quadratic module of A is a $\sum A^2-$module and the following holds.

Proposition 1.3.21. *Let M be a quadratic module of A. Then*
a) $H_M := \{a \in A : \exists\ N \in \mathbb{N} s.t.\ N \pm a \in M\}$ is a subring of A.
b) $M \cap H_M$ is an Archimedean quadratic module of H_M.
c) M is Archimedean if and only if $H_M = A$.
d) $\forall\ a \in A,\ a^2 \in H_M \Rightarrow a \in H_M$.
e) $\forall\ a_1, \ldots, a_k \in A,\ \sum_{i=1}^{k} a_i^2 \in H_M \Rightarrow a_i \in H_M\ \forall\ i = 1, \ldots, k.$

Proof. (see e.g. [23, Proposition 2.1, Lecture 28] and [29, Proposition 5.2.3]) □

Corollary 1.3.22. *Let M be a quadratic module of $\mathbb{R}[\underline{X}]$. The following are equivalent:*
(1) M is Archimedean
(2) $\exists\ N \in \mathbb{N}$ such that $N - \sum_{i=1}^{n} X_i^2 \in M$.
(3) $\exists\ N \in \mathbb{N}$ such that $N \pm X_i \in M$ for $i = 1, \ldots, n$.

Proof.
(1)⇒(2) This is clear from the definition of Archimedean quadratic module.
(2)⇒(3) Suppose that there exists $k \in \mathbb{N}$ such that $k - \sum_{i=1}^n X_i^2 \in M$. Then $\sum_{i=1}^n X_i^2 \in H_M$ and so Proposition 1.3.21-e) ensures that for each $i \in \{1, \ldots, n\}$ we have that $X_i \in H_M$. Hence, there exists $N \in \mathbb{N}$ such that $N \pm X_i \in M$, i.e. (3) holds.
(3)⇒(1) Suppose that there exists $k \in \mathbb{N}$ such that $k \pm X_i \in M$ for $i = 1, \ldots, n$. Then $X_i \in H_M$ for $i = 1, \ldots, n$ and since $\mathbb{R}^+ \subseteq M$ we also have that $\mathbb{R} \subseteq H_M$. Hence, Proposition 1.3.21-a) guarantees that $H_M = \mathbb{R}[\underline{X}]$, which is equivalent to the Archimedeanity of M by Proposition 1.3.21-c). □

For the general version of the Representation Theorem, we need to strengthen a bit our assumptions on T.

Definition 1.3.23. *A preprime T is said to be* weakly torsion *if for any $a \in A$ there exists a positive rational r and $m \in \mathbb{N}$ such that $(r+a)^m \in T$.*

Clearly, any Archimedean preprime is weakly torsion. Also, for $d \in \mathbb{N}$, any $2d-$power preordering of A is a weakly torsion preprime (just take $m = 2d$).

We are finally ready to state the general version of the Representation Theorem we had announced (for a proof see [27], [17] and [29, Theorem 5.4.4]). Other versions of the Representation Theorem can be found in [1], [19], [21, 22], [8], [39].

Theorem 1.3.24. *Let A be a commutative ring with 1 such that $\mathbb{Q} \subseteq A$. If T is a weakly torsion preprime of A and M an Archimedean $T-$module of A, then for any $a \in A$ we have:*

$$\hat{a} > 0 \text{ on } \mathcal{K}_M \Rightarrow a \in M.$$

Remark 1.3.25.

a) Taking $M = T$ with T Archimedean preprime, we get Krivine's version of the Representation Theorem (see [21, 22] and also [23, Corollary 2.1, Lecture 27]). From this version, we can already derive the Schmüdgen Positivstellensatz as it was first noted by Wörmann in [40] (see Theorem 1.3.31).

b) Taking $d \in \mathbb{N}$ and $T = \sum A^{2d}$, we get the Jacobi-Prestel Positivstellensatz (see Theorem 1.3.28) from which one can straightforwardly derive Putinar's Positivstellensatz (see Theorem 1.3.29).

To understand the meaning of the Representation Theorem 1.3.24 for $A = \mathbb{R}[X_1, \ldots, X_n]$, we need to understand what the Gelfand transform and the characters are in this special case.

Proposition 1.3.26.

a) The identity $id : \mathbb{R} \to \mathbb{R}$ is the unique ring homomorphism from $\mathbb{R}$ to $\mathbb{R}$

b) $X(\mathbb{R}[X_1, \ldots, X_n])$ is in a one-to-one correspondence with $\mathbb{R}^n$.

Proof.

a) Suppose that $\alpha : \mathbb{R} \to \mathbb{R}$ is a ring homomorphism such that $\alpha \neq id$. Then there exists $a \in \mathbb{R}$ such that $\alpha(a) \neq id(a)$, say $\alpha(a) < id(a)$. Thus, there exists $q \in \mathbb{Q}$ such that $\alpha(a) < q < id(a)$ and so $\alpha(a-q) < 0$ while $id(a-q) > 0$, i.e. $a - q \notin \alpha^{-1}(\mathbb{R}^+)$ and $a - q \in id^{-1}(\mathbb{R}^+)$. Hence, we have that $\alpha^{-1}(\mathbb{R}^+) \neq id^{-1}(\mathbb{R}^+)$. However, α maps squares to squares and so we also have that $\alpha^{-1}(\mathbb{R}^+) = \mathbb{R}^+ = id^{-1}(\mathbb{R}^+)$, which yields a contradiction.

b) By a), for any $\alpha \in X(\mathbb{R}[X_1, \ldots, X_n])$ we have that $\alpha \restriction_{\mathbb{R}} = id$, which easily implies that α is completely determined by $(\alpha(X_1), \ldots, \alpha(X_n)) \in \mathbb{R}^n$.

In fact, for any $p := \sum_\beta p_\beta \underline{X}^\beta \in \mathbb{R}[\underline{X}]$ with $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$ and $\underline{X}^\beta := X_1^{\beta_1} \cdots X_n^{\beta_n}$, we have that

$$\begin{aligned} \alpha(p) &= \alpha\left(\sum_\beta p_\beta \underline{X}^\beta\right) = \sum_\beta \alpha(p_\beta)\alpha(X_1)^{\beta_1} \cdots \alpha(X_n)^{\beta_n} \\ &= \sum_\beta p_\beta \alpha(X_1)^{\beta_1} \cdots \alpha(X_n)^{\beta_n} = p\left(\alpha(X_1), \dots, \alpha(X_n)\right). \end{aligned}$$

Conversely, for any $y \in \mathbb{R}^n$ we can define the map $\alpha_y : \mathbb{R}[X_1, \dots, X_n] \to \mathbb{R}$ by $\alpha_y(p) := p(y)$ for any $p \in \mathbb{R}[\underline{X}]$, which is clearly a ring homomorphism, i.e. $\alpha_y \in X(\mathbb{R}[X_1, \dots, X_n])$. Hence, $X(\mathbb{R}[X_1, \dots, X_n]) \cong \mathbb{R}^n$. $\square$

Remark 1.3.27. *Using the isomorphism between $X(\mathbb{R}[X_1, \dots, X_n])$ and $\mathbb{R}^n$ we get that for any $p \in \mathbb{R}[\underline{X}]$ the Gelfand transform $\hat{p}$ is identified with the polynomial p itself. Moreover, if M is a $2d-$power module of $\mathbb{R}[\underline{X}]$ then*

$$\begin{aligned} \mathcal{K}_M &= \{\alpha \in X(\mathbb{R}[X_1, \dots, X_n]) : \hat{q}(\alpha) \geq 0,\ \forall q \in M\} \\ &\cong \{x \in \mathbb{R}^n : q(x) \geq 0, \forall q \in M\} = K_M. \end{aligned}$$

In particular, if $S \subset \mathbb{R}[X_1, \dots, X_n]$ and M_S is the $2d-$power module generated by S then

$$\begin{aligned} \mathcal{K}_{M_S} \cong K_{M_S} &= \{x \in \mathbb{R}^n : q(x) \geq 0, \forall q \in M_S\} \\ &= \{x \in \mathbb{R}^n : q(x) \geq 0, \forall q \in S\} = K_S. \end{aligned}$$

Applying the Representation theorem 1.3.24 for $T = \sum \mathbb{R}[\underline{X}]^{2d}$ with $d \in \mathbb{N}$ and using Remark 1.3.27 we easily get the following results.

Theorem 1.3.28 (Jacobi-Prestel's Positivstellensatz)**.** *Let M be an Archimedean $2d-$power module of $\mathbb{R}[\underline{X}]$ with $d \in \mathbb{N}$. Then for any $f \in \mathbb{R}[\underline{X}]$*

$$f > 0 \ \text{on}\ K_M \Rightarrow f \in M.$$

In particular, taking $d = 1$ we easily get:

Theorem 1.3.29 (Putinar's Positivstellensatz)**.** *Let $S \subset \mathbb{R}[\underline{X}]$ be finite such that the quadratic module M_S generated by S is Archimedean. Then for any $f \in \mathbb{R}[\underline{X}]$*

$$f > 0 \ \text{on}\ K_S \Rightarrow f \in M_S.$$

To get Schmüdgen's Positivstellensatz from Theorem 1.3.24, we need to understand how the compactness of K_S relates to the Archimedeanity of the associated quadratic preordering T_S. The following criterion was provided by Wörmann in [40].

Theorem 1.3.30 (Wörmann Theorem)**.** *Let $S \subset \mathbb{R}[\underline{X}]$ be finite. The corresponding bcsas K_S is compact if and only if the associated quadratic preordering T_S is Archimedean.*

Proof. (see e.g. [29, Theorem 6.1.1] or [23, Theorem 2.1, Lecture 28]) □

Theorem 1.3.31 (Schmüdgen's Positivstellensatz)**.** *Let $S \subset \mathbb{R}[\underline{X}]$ be finite such that the associated bcsas K_S is compact. Then for any $f \in \mathbb{R}[\underline{X}]$*

$$f > 0 \text{ on } K_S \Rightarrow f \in T_S.$$

Proof. By Wörmann Theorem, the quadratic preordering T_S is Archimedean and so a weakly torsion preprime. Hence, by taking $T = M = T_S$ in the Representation Theorem 1.3.24 and using Remark 1.3.27, we obtain the conclusion. □