To get Schmüdgen's Positivstellensatz from Theorem 1.3.24, we need to understand how the compactness of K_S relates to the Archimedeanity of the associated quadratic preordering T_S. The following criterion was provided by Wörmann in [24].

Theorem 1.3.30 (Wörmann Theorem)**.** *Let $S \subset \mathbb{R}[\underline{X}]$ be finite. The corresponding bcsas K_S is compact if and only if the associated quadratic preordering T_S is Archimedean.*

Proof. (see e.g. [15, Theorem 6.1.1] or [12, Theorem 2.1, Lecture 28]) □

Theorem 1.3.31 (Schmüdgen's Positivstellensatz)**.** *Let $S \subset \mathbb{R}[\underline{X}]$ be finite such that the associated bcsas K_S is compact. Then for any $f \in \mathbb{R}[\underline{X}]$*

$$f > 0 \text{ on } K_S \Rightarrow f \in T_S.$$

Proof. By Wörmann Theorem, the quadratic preordering T_S is Archimedean and so a weakly torsion preprime. Hence, by taking $T = M = T_S$ in the Representation Theorem 1.3.24 and using Remark 1.3.27, we obtain the conclusion. □

Remark 1.3.32.

a) *Schmüdgen's Positivstellensatz fails in general if we drop the compactness assumption on K_S.*

   For example,

   - *for $n = 1$ and $S = \{X^3\}$, we have that $K_S = [0, \infty)$ is non-compact and $X + 1 > 0$ on K_S but $X + 1 \notin T_S$ (otherwise there would exist $\sigma_0, \sigma_1 \in \sum \mathbb{R}[X]^2$ such that $X + 1 = \sigma_0 + \sigma_1 X^3$ but this impossible as the right-hand side would have either even degree or odd degree ≥ 3 (see Example 1.3.6)).*
   - *for $n = 2$ and $S = \emptyset$, we have that the strictly positive version of the Motzkin polynomial $1 - X_1^2 X_2^2 + X_1^2 X_2^4 + X_1^4 X_2^2$ is indeed strictly positive on $K_S = \mathbb{R}^2$ but does not belong to $T_S = \sum \mathbb{R}[X_1, X_2]^2$.*

b) *Schmüdgen's Positivstellensatz fails in general if the assumption of strict positivity on K_S is replaced by the nonnegativity on K_S. For example, for $n = 1$ and $S = \{(1 - X^2)^3\}$ we have that $K_S = [-1, 1]$ is compact, $1 - X^2 \geq 0$ on K_S but $1 - X^2 \notin T_S$.*

c) *Schmüdgen's Positivstellensatz fails in general when the preordering T_S is replaced by the quadratic module M_S. The question of whether this was true was first posed by Putinar in [19] and got a negative answer in [8,* *Example 4.6], where Jacobi and Prestel showed that for* $n \geq 2$ *and* $S = \{g_1, \ldots, g_{n+1}\}$ *with* $g_i := X_i - \frac{1}{2}$ *for* $i = 1, \ldots, n$ *and* $g_{n+1} := 1 - \prod_{i=1}^n X_i$ *we have that* K_S *is compact but* M_S *is not Archimedean (thus, there exists* $N \in \mathbb{N}$ *such that* $N - \sum_{i=1}^n X_i^2 > 0$ *on* K_S *but* $N - \sum_{i=1}^n X_i^2 \notin M_S$*). This counterexample provides a general negative answer to Putinar's question, but there are actually cases in which the compactness of* K_S *implies the Archimedeanity of* M_S*. For instance, this holds in each of the following cases*

- $|S| = 1$ *(as in this case* $T_S = M_S$*)*
- $|S| = 2$ *(proof in [8]).*
- $n = 1$ *(proof in Sheet 2, Exercise 2)*
- S *consists only of linear polynomials (see [15, Theorem 7.1.3]).*

Note that if M_S *is Archimedean then* K_S *is always compact. Indeed, Archimedeanity of* M_S *implies that there exists* $N \in \mathbb{N}$ *such that* $N - \sum_{i=1}^n X_i^2 \in M_S$ *and so* $N - \sum_{i=1}^n X_i^2 \geq 0$ *on* K_S*. Hence,* K_S *is contained in the closed ball of radius* $\sqrt{N}$ *in* $\mathbb{R}^n$ *endowed with the euclidean topology, i.e.* K_S *is bounded. This together with the fact that* K_S *is closed provides that* K_S *is compact.*

Let us give now a further application of the Representaton Theorem 1.3.24, which shows the power of this very general version and allows to refine the representation provided by Putinar's Positivstellensatz (see Theorem 1.3.29).

Theorem 1.3.33. *Let* $S := \{g_1, \ldots, g_s\}$ *be a finite subset of* $\mathbb{R}[\underline{X}]$ *such that the associated quadratic module* M_S *is Archimedean. Then, for any real* $N > 0$*, any* $f > 0$ *on* K_S *can be represented as* $f = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s$ *where each* σ_i *is a sum of squares of polynomials which are strictly positive on the closed ball* $B_N := \{x \in \mathbb{R}^n : \|x\| \leq N\}$ *(here* $\|\cdot\|$ *is the euclidean norm).*

Proof. Let N be a strictly positive real number and $f > 0$ on K_S. Define

$$\tilde{T}^* := \{\sum f_i^2 : f_i \in \mathbb{R}[\underline{X}], f_i > 0 \text{ on } B_N\}, \tilde{T} := \tilde{T}^* \cup \{0\}$$

and

$$\tilde{M}^* := \tilde{T}^* + \tilde{T}^* g_1 + \cdots + \tilde{T}^* g_s, \tilde{M} := \tilde{M}^* \cup \{0\}.$$

As B_N is compact, for any $g \in \mathbb{R}[\underline{X}]$ there exists $r \in \mathbb{Q}$ positive such that $r + g > 0$ on B_N and so $(r+g)^2 \in \tilde{T}^*$. Hence, $\tilde{T}$ is a weakly torsion preprime.
<u>Claim</u>: For any $h \in \mathbb{R}[\underline{X}]$ there exists $l \in \mathbb{N}$ such that $l + h \in \tilde{M}^*$.
(see Sheet 2, Exercise 3 for a proof of the Claim).

Since $\tilde{T}$ is a preprime, it easily follows from the definitions that $\tilde{M} + \tilde{M} \subseteq \tilde{M}$ and $\tilde{T}\tilde{M} \subseteq \tilde{M}$. Moreover, applying the claim for $h = 0$, we have that there exists $l \in \mathbb{N}$ such that $l \in \tilde{M}^*$ and so $1 = l \cdot \frac{1}{l} \in \mathbb{Q}\tilde{M}^* \subseteq \tilde{M}^* \subseteq \tilde{M}$. Thus, $\tilde{M}$ is a $\tilde{T}-$module. By the Claim, $\tilde{M}$ is also Archimedean.

To apply Theorem 1.3.24, it remains to show that $K_S = K_{\tilde{M}}$. Once this is proved, the theorem ensures that $f \in \tilde{M}$.

($\subseteq$) As $\tilde{M} \subseteq M_S$, we have that $K_S \subseteq \{x \in \mathbb{R}^n : g(x) \geq 0, \forall g \in \tilde{M}\} = K_{\tilde{M}}$.

($\supseteq$) Suppose there exists $x \in K_{\tilde{M}}$ such that $x \notin K_S$. Then there exists $i \in \{1, \ldots, s\}$ such that $g_i(x) < 0$. Take $h := \sum_{j=0}^{s} r_j g_j$ with $g_0 := 1$, $r_j = 1$ for all $j \neq i$, and $r_i > ls$ where $l \in \mathbb{N}$ such that $g_j(x) < -lg_i(x)$ for all $j \neq i$. Thus, $h \in \tilde{M}$ but $h(x) = \sum_{j \neq i} g_j(x) + r_i g_i(x) < -lsg_i(x) + r_i g_i(x) = (r_i - ls)g_i(x) < 0$, which yields $x \notin K_{\tilde{M}}$ that is a contradiction. □

1.3.3 Closure of even power modules

In this section, we are going to see how the Positivstellensätze considered in the previous section can be used to better understand the relation between Psd(K_S) and T_S (resp. M_S). For this purpose, let us recall the following application of Hahn-Banach Theorem which we have studied in [6, Section 5.2].

Corollary 1.3.34. *Let (X, τ) be a locally convex t.v.s. over the real numbers. If C is a nonempty closed cone of X and x and $x_0 \in X \setminus C$, then there exists a linear $\tau-$continuous functional $L : X \to \mathbb{R}$ non identically zero s.t. $L(C) \geq 0$ and $L(x_0) < 0$.*

Recall that a cone of X is a subset $C \subseteq X$ such that $C + C \subseteq C$ and $\lambda C \subseteq C$ for all $\lambda \in \mathbb{R}^+$.

Proof. As C is closed in (X, τ) and $x_0 \in X \setminus C$, we have that $X \setminus C$ is an open neighbourhood of x_0. Then the local convexity of (X, τ) guarantees that there exists an open convex neighbourhood V of x_0 s.t. $V \subseteq X \setminus C$ i.e. $V \cap C = \emptyset$. By the Geometric form of Hahn-Banach theorem, we have that there exists a closed hyperplane H of X separating V and C, i.e. there exists $L : X \to \mathbb{R}$ linear $\tau-$continuous and not identically zero s.t. $L(C) \geq 0$ and $L(V) < 0$ (see [6, Proposition 5.2.1-c)] for more details). In particular, $L(C) \geq 0$ and $L(x_0) < 0$. □

Given a convex cone C in any t.v.s. (X, τ) we define the first and the second dual of C w.r.t. τ respectively as follows:

$$C_\tau^\vee := \{\ell : X \to \mathbb{R} \text{ linear} \,|\, \ell \text{ is } \tau-\text{continuous and } \ell(C) \geq 0\}$$

$$C_\tau^{\vee\vee} := \{x \in X \,|\, \forall\, \ell \in C_\tau^\vee,\, \ell(x) \geq 0\}.$$

Note that

- $C \subseteq C_\tau^{\vee\vee}$, because if $x \in C$ then for all $\ell \in C_\tau^\vee$ we have $\ell(x) \geq 0$ by definition of $C_\tau^\vee$.
- $C_\tau^{\vee\vee}$ is closed in (X, τ), because $C_\tau^{\vee\vee} = \bigcap_{\ell \in C_\tau^\vee} \ell^{-1}([0,\infty))$ and each $\ell \in C_\tau^\vee$ is by definition $\tau-$continuous.

Hence, $\overline{C}^\tau \subseteq C_\tau^{\vee\vee}$ always holds.

Corollary 1.3.35. *Let (X, τ) be a locally convex t.v.s. over the real numbers. If C is a nonempty convex cone in X, then $\overline{C}^\tau = C_\tau^{\vee\vee}$.*

Proof. Suppose there exists $x_0 \in C_\tau^{\vee\vee} \setminus \overline{C}^\tau$. By Corollary 1.3.34, there exists a $\tau-$continuous functional $L : X \to \mathbb{R}$ non identically zero s.t. $L(\overline{C}^\tau) \geq 0$ and $L(x_0) < 0$. As $L(C) \geq 0$ and L is $\tau-$continuous, we have $L \in C_\tau^\vee$. This together with the fact that $L(x_0) < 0$ give $x_0 \notin C_\tau^{\vee\vee}$, which is a contradiction. Hence, $\overline{C}^\tau = C_\tau^{\vee\vee}$. □

The previous results clearly apply to $\mathbb{R}[\underline{X}]$ endowed with the finite topology τ_f. Indeed, we have already observed in Section 1.1 that τ_f is actually the finest locally convex topology on $\mathbb{R}[\underline{X}]$ and so that $(\mathbb{R}[\underline{X}], \tau f)$ is a locally convex t.v.s.. Moreover, keeping in mind [21, Theorem 3.1.1], it is easy to prove that $(\mathbb{R}[\underline{X}], \tau_f)$ is actually a topological algebra, i.e. a t.v.s. with separately continuous multiplication. Hence, we can prove the following properties.

Proposition 1.3.36. *Let $d \in \mathbb{N}$, M a $2d-$power module of $\mathbb{R}[\underline{X}]$ and φ the finest locally convex topology on $\mathbb{R}[\underline{X}]$. Then*
(a) $\overline{M}^\varphi$ is a $2d-$power module of $\mathbb{R}[\underline{X}]$
(b) If M is a preordering, then $\overline{M}^\varphi$ is a preordering.
(c) $\overline{M}^\varphi = M_\varphi^{\vee\vee} \subseteq \mathrm{Psd}(K_M)$

Proof. (a) As M is a $2d-$power module of $\mathbb{R}[\underline{X}]$ and $(\mathbb{R}[\underline{X}], \varphi)$ is a topological algebra, we have that $1 \in M \subseteq \overline{M}^\varphi$, $\overline{M}^\varphi + \overline{M}^\varphi \subseteq \overline{M + M}^\varphi \subseteq \overline{M}^\varphi$ and $p^{2d}\overline{M}^\varphi \subseteq \overline{p^{2d}M}^\varphi \subseteq \overline{M}^\varphi$. Hence, $\overline{M}^\varphi$ is a $2d-$power module.

(b) If M is a $2d-$power preordering, then (a) ensures that $\overline{M}^\varphi$ is a $2d-$power module. Moreover, using that $M \cdot M \subseteq M$ and $(\mathbb{R}[\underline{X}], \varphi)$ is a topological algebra, we get that $\overline{M}^\varphi \cdot \overline{M}^\varphi \subseteq \overline{M \cdot M}^\varphi \subseteq \overline{M}^\varphi$. Hence, $\overline{M}^\varphi$ is a preordering.

(c) Since every $2d-$power module is a cone, Corollary 1.3.35 guarantees that $\overline{M}^\varphi = M_\varphi^{\vee\vee}$. For any $x \in \mathbb{R}^n$, the map $e_x : \mathbb{R}[\underline{X}] \to \mathbb{R}$ defined by $e_x(p) := p(x)$ is a $\mathbb{R}-$algebra homomorphism. Hence, for all $x \in \mathbb{R}^n$, e_x is linear and so $\varphi-$continuous. Also, for all $x \in K_M$, we have that $e_x(g) = g(x) \geq 0$ for all $g \in M$, i.e. $e_x \in M_\varphi^\vee$. Then for any $f \in M_\varphi^{\vee\vee}$ we get that $f(x) = e_x(f) \geq 0$ for all $x \in K_M$, i.e. $f \in \mathrm{Psd}(K_S)$. □

Let us now come back to the Positivstellensätze introduced in the last sections and derive from them the corresponding Nichtnegativstellensätze.

Corollary 1.3.37 (Jacobi-Prestel's Nichtnegativstellensatz)**.** *Let M be an Archimedean $2d-$power module of $\mathbb{R}[\underline{X}]$ with $d \in \mathbb{N}$. Then for any $f \in \mathbb{R}[\underline{X}]$*

$$f \geq 0 \text{ on } K_M \Rightarrow \forall \varepsilon > 0, f + \varepsilon \in M.$$

Proof. Let $f \in \mathbb{R}[\underline{X}]$ be such that $f \geq 0$ on K_M. Then for any $\varepsilon > 0$, we have that $f + \varepsilon > 0$ on K_M and so Theorem 1.3.28 ensures that $f + \varepsilon \in M$. □

Corollary 1.3.38 (Putinar's Nichtnegativstellensatz)**.** *Let $S \subset \mathbb{R}[\underline{X}]$ be finite such that the quadratic module M_S generated by S is Archimedean. Then for any $f \in \mathbb{R}[\underline{X}]$*

$$f \geq 0 \text{ on } K_S \Rightarrow \forall \varepsilon > 0, f + \varepsilon \in M_S.$$

Corollary 1.3.39 (Schmüdgen's Nichtnegativstellensatz)**.** *Let $S \subset \mathbb{R}[\underline{X}]$ be finite such that the associated bcsas K_S is compact. Then for any $f \in \mathbb{R}[\underline{X}]$*

$$f \geq 0 \text{ on } K_S \Rightarrow \forall \varepsilon > 0, f + \varepsilon \in T_S.$$

Using Proposition 1.3.36 and the Nichtnegativstellensätze, we easily get the following closure results.

Corollary 1.3.40. *Let M be an Archimedean $2d-$power module of $\mathbb{R}[\underline{X}]$ with $d \in \mathbb{N}$. Then $\mathrm{Psd}(K_M) = \overline{M}^{\varphi}$.*

Proof. By Proposition 1.3.36-(c), $\mathrm{Psd}(K_M) \supseteq \overline{M}^{\varphi}$. For the converse inclusion, let $f \in \mathrm{Psd}(K_M)$ and $\varepsilon > 0$. The Jacobi-Prestel's Nichtnegativstellensatz 1.3.37 guarantees that $f + \varepsilon \in M$ and so, for any $\ell \in M_{\varphi}^{\vee}$, we have that $\ell(f + \varepsilon) \geq 0$, i.e. $\ell(f) \geq -\varepsilon \ell(1)$. Then $\ell(f) \geq 0$ and so $f \in M_{\varphi}^{\vee\vee} \overset{\text{Cor1.3.35}}{=} \overline{M}^{\varphi}$. □

Corollary 1.3.41. *Let $S \subset \mathbb{R}[\underline{X}]$ be finite such that the quadratic module M_S generated by S is Archimedean. Then $\mathrm{Psd}(K_S) = \overline{(M_S)}^{\varphi}$*

Corollary 1.3.42. *Let $S \subset \mathbb{R}[\underline{X}]$ be finite such that the associated bcsas K_S is compact. Then $\mathrm{Psd}(K_S) = \overline{(T_S)}^{\varphi}$*

These results make us understanding that even when we do not have saturation of the preordering we still have cases when $\mathrm{Psd}(K_S)$ can be characterized in terms of T_S or M_S, namely as closures of these cones w.r.t. the finest locally convex topology φ. Note that typically T_S is not closed.

In fact, if S is a finite subset of $\mathbb{R}[\underline{X}]$ such that K_S is compact and $\dim(K_S) \geq 3$, then Corollary 1.3.42 ensures that $\mathrm{Psd}(K_S) = \overline{(T_S)}^{\varphi}$ but by Theorem 1.3.5 we also know that $\mathrm{Psd}(K_S) \neq T_S$ so $T_S \neq \overline{(T_S)}^{\varphi}$, i.e.$T_S$ is not closed in $(\mathbb{R}[\underline{X}], \varphi)$.

In the case when K_S is not compact (and so M_S is not Archimedean), we cannot apply the previous closure results so is it natural to ask if we can get similar results by considering closures w.r.t. other locally convex topologies.