In fact, if $S$ is a finite subset of $\mathbb{R}[\underline{X}]$ such that $K_S$ is compact and $\dim(K_S) \geq 3$, then Corollary 1.3.42 ensures that $\mathrm{Psd}(K_S) = \overline{(T_S)}^{\varphi}$ but by Theorem 1.3.5 we also know that $\mathrm{Psd}(K_S) \neq T_S$ so $T_S \neq \overline{(T_S)}^{\varphi}$, i.e.$T_S$ is not closed in $(\mathbb{R}[\underline{X}], \varphi)$. In the case when $K_S$ is not compact (and so $M_S$ is not Archimedean), we cannot apply the previous closure results, so is it natural to ask if we can get similar results by considering closures w.r.t. other topologies rather than $\varphi$.

Closures of even power modules of $\mathbb{R}[X_1, \ldots, X_n]$ have been studied already since the seventies. Indeed, the cone $\sum \mathbb{R}[X_1, \ldots, X_n]^2$ is closed in $(\mathbb{R}[\underline{X}], \varphi)$ (see Sheet 3, Exercise 2), so taking its closure w.r.t. $\varphi$ does not help to characterize $\mathrm{Psd}(\mathbb{R}^n)$ for $n \geq 2$ (as $\mathrm{Psd}(\mathbb{R}^n) \neq \sum \mathbb{R}[\underline{X}]^2 = \overline{\sum \mathbb{R}[\underline{X}]^2}^{\varphi}$). However, every polynomial in $\mathrm{Psd}(\mathbb{R}^n)$ can be approximated by elements in $\sum \mathbb{R}[\underline{X}]^2$ w.r.t. the topology induced by the norm $\|\cdot\|_1$, where $\|f\|_1 := \sum_\alpha |f_\alpha|$ for any $f = \sum_\alpha f_\alpha \underline{X}^\alpha \in \mathbb{R}[\underline{X}]$. In fact, in [2, Theorem 9.1] the authors show that $\mathrm{Psd}([-1,1]^n) = \overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_1}$, i.e. $\sum \mathbb{R}[\underline{X}]^2$ is dense in $\mathrm{Psd}([-1,1]^n)$ w.r.t. $\|\cdot\|_1$ on $\mathbb{R}[\underline{X}]$ (see also [23]). This result is actually established in [2] as a corollary of a general result valid for any commutative semigroup. In [3] and [4] the results in [2] were extended further, to include commutative semigroups with involution and topologies induced by absolute values. In [11] a new proof of these results is given by using the Representation Theorem 1.3.24 and they are at the same time extended from sums of squares to sums of $2d-$powers. In particular, the authors prove that for any $d \in \mathbb{N}$ we get $\mathrm{Psd}([-1,1]^n) = \overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\|\cdot\|_1}$. The closure of $\sum \mathbb{R}[\underline{X}]^{2d}$ w.r.t. to $\|\cdot\|_p$ with $1 \leq p \leq \infty$ has been studied in [9], where it is showed that for any $d \in \mathbb{N}$ we have $\mathrm{Psd}([-1,1]^n) = \overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\|\cdot\|_p}$. In this same work also the closure of $\sum \mathbb{R}[\underline{X}]^{2d}$ w.r.t. weighted versions of $\|\cdot\|_p$ has been considered. In particular, Lasserre in [22] identified a weighted version $\|\cdot\|_w$ of the norm $\|\cdot\|_1$ such that for any $S \subseteq \mathbb{R}[\underline{X}]$ finite $\mathrm{Psd}(K_S) = \overline{M_S}^{\|\cdot\|_w}$.

The question of establishing when the closure of an even power module $M$ in $\mathbb{R}[\underline{X}]$ coincides with $\mathrm{Psd}(K)$ for some subset $K$ of $\mathbb{R}^n$ can be clearly considered also for even power modules in any unital commutative topological $\mathbb{R}-$algebra. Such a general setting was studied in [10] and [12]. We would like to present here the main result [12] as it is a powerful application of the Representation Theorem 1.3.24 and allows to deduce several of the closure results mentioned above.

Let $A$ be a unital commutative $\mathbb{R}-$algebra and denote by $X(A)$ the character space of $A$ (see Section 1.3.2 for the definition). For any $M \subseteq A$, recall that $\mathcal{K}_M := \{\alpha \in X(A) : \hat{a}(\alpha) \geq 0, \forall a \in M\}$, where $\hat{a}$ is the Gelfand transform of $a$ (see Section 1.3.2 for the definition).

**Definition 1.3.43.** *A function $\rho : A \to \mathbb{R}$ is called a* seminorm *if*

1. *$\rho$ is subadditive: $\forall x, y \in A$, $\rho(x+y) \leq \rho(x) + \rho(y)$.*
2. *$\rho$ is positively homogeneous: $\forall x \in A$, $\forall \lambda \in \mathbb{R}, \rho(\lambda x) = |\lambda|\rho(x)$.*

*A seminorm on a $A$ is said to be* submultiplicative *if*

$$\forall x, y \in A,\ \rho(xy) \leq \rho(x)\rho(y).$$

If $\rho$ is a submultiplicative seminorm on $A$, then $(A, \rho)$ is called a *seminormed algebra*. (In particular, $A$ with a submultiplicative norm is said to be a normed algebra). Note that any seminormed algebra is a topological algebra with jointly continuous multiplication (c.f. [14, Proposition 1.2.14]). We denote by $\mathfrak{sp}(\rho)$ the set of all $\rho-$continuous $\mathbb{R}-$algebra homomorphisms from $A$ to $\mathbb{R}$ and we refer to $\mathfrak{sp}(\rho)$ as the *Gelfand spectrum* of $(A, \rho)$, i.e.

$$\mathfrak{sp}(\rho) := \{\alpha \in \underline{X}(A) : \alpha \text{ is } \rho - \text{continuous}\}.$$

**Lemma 1.3.44.**
*For any seminormed $\mathbb{R}-$algebra $(A, \rho)$ we have:*

$$\mathfrak{sp}(\rho) = \{\alpha \in \underline{X}(A) : |\alpha(a)| \leq \rho(a) \text{ for all } a \in A\}.$$

*Proof.* The inclusion $\{\alpha \in \underline{X}(A) : |\alpha(a)| \leq \rho(a) \text{ for all } a \in A\} \subseteq \mathfrak{sp}(\rho)$ follows straightforward from the definition of Gelfand spectrum of $(A, \rho)$. Let us prove by contradiction the converse inclusion.

Suppose that $\alpha \in \underline{X}(A)$ is $\rho-$continuous but that there exists $x \in A$ s.t. $|\alpha(x)| > \rho(x)$. Take $\delta \in \mathbb{R}^+$ s.t. $|\alpha(x)| > \delta > \rho(x)$ and set $y := \frac{x}{\delta}$. Then we have $\rho(y) < 1$ and $|\alpha(y)| > 1$, which imply that $\rho(y^n) \to 0$ and $|\alpha(y^n)| \to \infty$ as $n \to \infty$, contradicting the $\rho-$continuity of $\alpha$. □

We are ready now to state the main result of [12].

**Theorem 1.3.45.** *Let $(A, \rho)$ be a unital commutative seminormed $\mathbb{R}-$algebra and $d \in \mathbb{N}$. If $M$ is a $2d-$power module of $A$, then $\overline{M}^{\rho} = \mathrm{Psd}(\mathcal{K}_M \cap \mathfrak{sp}(\rho))$.*

In order to prove this result, let us recall some fundamental properties of unital commutative seminormed $\mathbb{R}$-algebras.

**Remark 1.3.46.** *Any seminormed algebra $(A, \rho)$ can be topologically embedded into a Banach algebra $(\tilde{A}, \tilde{\rho})$, i.e. there exists $\iota : (A, \rho) \to (\tilde{A}, \tilde{\rho})$ continuous embedding (see [14, Corollary 3.3.21]). Hence, $A$ and $\iota(A)$ are homeomorphic. Recall that a Banach algebra is a normed algebra whose underlying space is a complete normed space.*

**Lemma 1.3.47.** *For any unital commutative Banach $\mathbb{R}-$algebra $(B, \sigma)$, any $b \in B$ and $r \in \mathbb{R}$ such that $r > \sigma(b)$, and any $k \in \mathbb{N}$, there exists $p \in B$ such that $p^k = r + b$.*

*Proof.* The standard power series expansion

$$(r+x)^{\frac{1}{k}} = r^{\frac{1}{k}}\left(1+\frac{x}{r}\right)^{\frac{1}{k}} = r^{\frac{1}{k}}\sum_{j=0}^{\infty}\frac{\frac{1}{k}\left(\frac{1}{k}-1\right)\cdots\left(\frac{1}{k}-j\right)}{j!}\left(\frac{x}{r}\right)^{j}$$

converges absolutely for $|x| < r$. This together with the fact that $(B, \sigma)$ is a Banach algebra implies that, for any $b \in B$ and any $r \in \mathbb{R}$ such that $r > \sigma(b)$, we have

$$p := r^{\frac{1}{k}}\sum_{j=0}^{\infty}\frac{\frac{1}{k}\left(\frac{1}{k}-1\right)\cdots\left(\frac{1}{k}-j\right)}{j!}\left(\frac{b}{r}\right)^{j} \in B$$

and $p^k = (r + b)$. □

**Lemma 1.3.48.** *Let $(B, \sigma)$ be a unital Banach $\mathbb{R}$-algebra and $L : B \to \mathbb{R}$ a linear functional. If there exists $d \in \mathbb{N}$ such that $L(b^{2d}) \geq 0$ for all $b \in B$, then $L$ is $\sigma-$continuous.*

*Proof.* By Lemma 1.3.47, for all $n \in \mathbb{N}$ and all $a \in B$ we have that $\frac{1}{n} + \sigma(a) \pm a = 1 + \sigma(\pm a) + (\pm a) \in B^{2d}$. Applying $L$, we obtain $|L(a)| \leq (\frac{1}{n} + \sigma(a))L(1)$ for all $n \in \mathbb{N}$ and all $a \in B$, so $|L(a)| \leq \sigma(a)L(1)$ for all $a \in B$. Hence, $L$ is $\sigma-$continuous. □

We are finally ready to show Theorem 1.3.45.

*Proof. of Theorem 1.3.45*
Since

$$\text{Psd}(\mathcal{K}_M \cap \mathfrak{sp}(\rho)) = \bigcap_{\alpha \in \mathcal{K}_M \cap \mathfrak{sp}(\rho)} \alpha^{-1}([0, +\infty))$$

and any $\alpha \in \mathcal{K}_M \cap \mathfrak{sp}(\rho)$ is $\rho-$continuous, we have that $\text{Psd}(\mathcal{K}_M \cap \mathfrak{sp}(\rho))$ is closed in $(A, \rho)$. Hence, $\overline{M}^{\rho} \subseteq \overline{\text{Psd}(\mathcal{K}_M \cap \mathfrak{sp}(\rho))}^{\rho} = \text{Psd}(\mathcal{K}_M \cap \mathfrak{sp}(\rho))$. For the reverse inclusion, let us consider $b \in \text{Psd}(\mathcal{K}_M \cap \mathfrak{sp}(\rho))$ and denote by $\widetilde{M}$ the closure of the image of $M$ in the Banach algebra $(\tilde{A}, \tilde{\rho})$, i.e. $\widetilde{M} := \overline{\iota(M)}^{\tilde{\rho}}$ (see Remark 1.3.46). Then $\widetilde{M}$ is a $2d-$power module of $\tilde{A}$ as addition and multiplication on $\tilde{A}$ are both $\tilde{\rho}$-continuous and $M$ is a $2d-$power module of $A$. By Lemma 1.3.47, for any $n \in \mathbb{N}$ and all $a \in \tilde{A}$ we have $\frac{1}{n}+\tilde{\rho}(a)\pm a \in \tilde{A}^{2d} \subseteq \widetilde{M}$. Hence, $\tilde{\rho}(a) \pm a \in \widetilde{M}$ which implies that $\widetilde{M}$ is Archimedean. Now, for any $\alpha \in \mathcal{K}_{\widetilde{M}}$ we have that $\alpha(a) \geq 0$ for all $a \in \widetilde{M}$, which gives in particular that $\alpha$ is a linear functional on $\tilde{A}$ s.t. $\alpha(a^{2d}) \geq 0$ for all $a \in \tilde{A}$ and so Lemma 1.3.48 ensures that $\alpha$ is $\tilde{\rho}-$continuous. Hence, $\alpha \circ \iota$ is $\rho-$continuous and $\alpha(\iota(m)) \geq 0$ for all $m \in M$, i.e.

$$(\alpha \circ \iota) \in \mathcal{K}_M \cap \mathfrak{sp}(\rho), \forall \alpha \in \mathcal{K}_{\widetilde{M}}. \tag{1.5}$$

Denote by $\tilde{b} := \iota(b)$. Then (1.5) ensures that for all $\alpha \in \mathcal{K}_{\widetilde{M}}$ we have $\alpha(\tilde{b}) = (\alpha \circ \iota)(b) \geq 0$ as by assumption $b \in \text{Psd}(\mathcal{K}_M \cap \mathfrak{sp}(\rho))$. By Jacobi-Prestel Nichnegativstellensatz we have that for all $n \in \mathbb{N}$, $\tilde{b} + \frac{1}{n} \in \widetilde{M}$ and so by the completeness of $\tilde{A}$ we get $\tilde{b} \in \widetilde{M}$. This yields $\iota(b) \in \overline{\iota(M)}^{\tilde{\rho}} = \iota(\overline{M}^{\rho})$ where the latter equality holds since $A$ and $\iota(A)$ are homeomorphic (see Remark 1.3.46). Hence, $b \in \overline{M}^{\rho}$. □

Keeping in mind the identification between $X(\mathbb{R}[\underline{X}])$ and $\mathbb{R}^n$ proved in Proposition 1.3.26 and applying Theorem 1.3.45 for $A = \mathbb{R}[\underline{X}]$, we obtain some of the closure results mentioned above.

**Examples 1.3.49.** *Let $M := \sum \mathbb{R}[\underline{X}]^2$ and so $K_M = \mathbb{R}^n$.*

(a) *If we consider the norm $\|\cdot\|_1$ defined by $\|f\|_1 := \sum_\beta |f_\beta|$ for all $f = \sum_\beta f_\beta \underline{X}^\beta \in \mathbb{R}[\underline{X}]$, then $(\mathbb{R}[\underline{X}], \|\cdot\|_1)$ is a normed algebra. Hence, Theorem 1.3.45 gives $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_1} = \text{Psd}(\mathbb{R}^n \cap \mathfrak{sp}(\|\cdot\|_1))$. Let us now compute the Gelfand spectrum of $(\mathbb{R}[\underline{X}], \|\cdot\|_1)$.*
*If $y = (y_1, \ldots, y_n) \in \mathfrak{sp}(\|\cdot\|_1)$, then by Lemma 1.3.44 we obtain that $|p(y)| \leq \|p\|_1$ for all $p \in \mathbb{R}[\underline{X}]$ and in particular for each $i = 1, \ldots, n$ we have $|y_i| \leq \|X_i\|_1 = 1$. Hence, $y \in [-1,1]^n$. Conversely, for any $y = (y_1, \ldots, y_n) \in [-1,1]^n$ we have that $|y_i| = 1$ for $i = 1, \ldots, n$ and so for any $p = \sum_\beta p_\beta \underline{X}^\beta \in \mathbb{R}[\underline{X}]$ we get*

$$|p(y)| \leq \sum_\beta |p_\beta||y_1|^{\beta_1} \cdot |y_n|^{\beta_n} \leq \sum_\beta |p_\beta| = \|p\|_1.$$

*Hence, by Lemma 1.3.44, $y \in \mathfrak{sp}(\|\cdot\|_1)$.*

*We have therefore showed that $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_1} = \text{Psd}([-1,1]^n)$, retrieving the result of [2] and [23].*

*(b) Let* $1 \leq p < \infty$ *and consider* $\|\cdot\|_p$*, where for any* $f = \sum_\beta f_\beta \underline{X}^\beta \in \mathbb{R}[\underline{X}]$ *we define* $\|f\|_p := \left(\sum_\beta |f_\beta|^p\right)^{\frac{1}{p}}$ *for* $1 \leq p < \infty$ *and* $\|f\|_\infty := \max_\beta |f_\beta|$*. As* $\|f\|_p \leq \|f\|_1$ *for all* $f \in \mathbb{R}[\underline{X}]$*, we have that* $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_1} \subseteq \overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_p}$ *and so by (a) we obtain* $\mathrm{Psd}([-1,1]^n) \subseteq \overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_p}$*. Furthermore, for any* $y \in [-1,1]^n$ *we have that the map* $e_y : \mathbb{R}[\underline{X}] \to \mathbb{R}$*, defined by* $e_y(f) := f(y)$ *for any* $f \in \mathbb{R}[\underline{X}]$*, is* $\|\cdot\|_p-$*continuous. Indeed, for any* $y = (y_1, \ldots, y_n) \in [-1,1]^n$ *and any* $f = \sum_\beta f_\beta \underline{X}^\beta \in \mathbb{R}[\underline{X}]$ *we have that*

$$|e_y(f)| = |f(y)| \leq \sum_\beta |f_\beta||y_1|^{\beta_1} \cdots |y_n|^{\beta_n} \overset{\text{Hölder ineq.}}{\leq} C_q \|f\|_p,$$

*where* $1 \leq q \leq \infty$ *is such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and*

$$C_q := \begin{cases} \left(\sum_\beta |y_1|^{q\beta_1} \cdots |y_n|^{q\beta_n}\right)^{\frac{1}{q}} & \text{if } q < \infty \\ \max_\beta |y_1|^{\beta_1} \cdots |y_n|^{\beta_n} & \text{if } q = \infty \end{cases}$$

*which is finite as* $y = (y_1, \ldots, y_n) \in [-1,1]^n$*.*

*Hence,* $\mathrm{Psd}([-1,1]^n) = \bigcap_{y \in [-1,1]^n} e_y^{-1}([0, +\infty))$ *is closed in* $(\mathbb{R}[\underline{X}], \|\cdot\|_p)$*, which yields* $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_p} \subseteq \overline{\mathrm{Psd}([-1,1]^n)}^{\|\cdot\|_p} = \mathrm{Psd}([-1,1]^n)$*. We have therefore showed that* $\mathrm{Psd}([-1,1]^n) = \overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_p}$*, for all* $1 \leq p \leq \infty$*, retrieving the result of [9].*

Theorem 1.3.45 easily extends to locally multiplicatively convex algebras.

**Definition 1.3.50.** *A unital commutative* $\mathbb{R}-$*algebra* $A$ *endowed with a locally convex topology induced by a family of submultiplicative seminorms on* $A$ *is called* locally multiplicatively convex (lmc)*.*

If $(A, \tau)$ is an lmc algebra, then it is a topological algebra with jointly continuous multiplication (c.f. [14, Proposition 2.1.9]). Moreover, we denote by $\mathfrak{sp}(\tau)$ the set of all $\tau-$continuous $\mathbb{R}-$algebra homomorphisms from $A$ to $\mathbb{R}$ and we refer to $\mathfrak{sp}(\tau)$ as the *Gelfand spectrum* of $(A, \tau)$.

Using that any locally convex topology can be always generated by a family of directed seminorms (see [13, Proposition 4.2.14]) we get the following result.

**Proposition 1.3.51.** *Let* $(A, \tau)$ *be an lmc algebra with* $\tau$ *generated by a directed family* $\mathcal{F}$ *of submultiplicative seminorms. Then* $\mathfrak{sp}(\tau) = \bigcup_{\rho \in \mathcal{F}} \mathfrak{sp}(\rho)$*.*

*Proof.* Applying [13, Proposition 4.6.1] and the definition of Gelfand spectrum, we easily obtain

$$\begin{aligned} \mathfrak{sp}(\tau) &= \{\alpha \in X(A) : \alpha \text{ is } \tau\text{−continuous}\} \\ &= \bigcup_{\rho \in \mathcal{F}} \{\alpha \in X(A) : \alpha \text{ is } \rho\text{−continuous}\} = \bigcup_{\rho \in \mathcal{F}} \mathfrak{sp}(\rho). \end{aligned}$$

□

It is then clear how to extend Theorem 1.3.45 to any lmc algebra.

**Theorem 1.3.52.** *Let $(A, \tau)$ be an lmc algebra and $d \in \mathbb{N}$. If $M$ is a $2d$−power module of $A$, then $\overline{M}^{\tau} = \mathrm{Psd}(\mathcal{K}_M \cap \mathfrak{sp}(\tau))$.*

*Proof.* Let $\mathcal{F}$ be a directed family of submultiplicative seminorms generating $\tau$. Then by Proposition 1.3.51, we get

$$\begin{aligned} \overline{M}^{\tau} &= \bigcap_{\rho \in \mathcal{F}} \overline{M}^{\rho} = \bigcap_{\rho \in \mathcal{F}} \mathrm{Psd}\left(\mathcal{K}_M \cap \mathfrak{sp}(\rho)\right) \\ &= \mathrm{Psd}\left(\mathcal{K}_M \cap \bigcup_{\rho \in \mathcal{F}} \mathfrak{sp}(\rho)\right) = \mathrm{Psd}\left(\mathcal{K}_M \cap \mathfrak{sp}(\tau)\right). \end{aligned}$$

□