Chapter 2

$K-$Moment Problem: formulation and connection to $\mathrm{Psd}(K)$

2.1 Formulation: from finite to infinite dimensional settings.

As suggested by the name the $K-$Moment Problem deals with moments of measures. In this course we are going to consider always *non-negative Radon measures* on Hausdorff topological spaces.

Recall that

Definition 2.1.1. *A* Radon measure μ *on a Hausdorff space* (X, τ) *is a measure defined on the Borel* $\sigma-$*algebra* $\mathcal{B}_\tau$ *on* (X, τ) *(i.e. the smallest* $\sigma-$*algebra on* X *containing* τ*) and such that*

- μ *is locally finite, i.e. for all* $x \in X$ *there exists* U *open neighbourhood of* x *in* (X, τ) *such that* $\mu(U) < \infty$*)*
- μ *is inner regular, i.e. for all* $B \in \mathcal{B}_\tau$, $\mu(B) = \sup\{\mu(K) : K \subseteq B \, compact\}$.

We say that μ *is* supported in *a subset* Y *of* X *if for any* $B \in \mathcal{B}_\tau$ *we have that* $B \cap Y = \emptyset$ *implies* $\mu(B) = 0$.

Let us start by introducing the most classical version of the $K-$moment problem.

Given a Radon measure μ on $\mathbb{R}$ and $j \in \mathbb{N}_0$, the $j-th$ *moment of* μ is defined as

$$m_j^\mu := \int_{\mathbb{R}} x^j \mu(dx).$$

If all moments of μ exist and are finite, then we can associate to μ the sequence of real numbers $(m_j^\mu)_{j \in \mathbb{N}_0}$, which is said to be the *moment sequence* of μ. The moment problem exactly addresses the inverse question:

Problem 2.1.2 (The one-dimensional $K-$Moment Problem (KMP))**.**
Let $N \in \mathbb{N}_0 \cup +\infty$. Given a closed subset K of $\mathbb{R}$ and a sequence $m := (m_j)_{j=0}^N$ of real numbers, does there exist a non-negative Radon measure μ supported in K and s.t. $m_j = m_j^\mu$ for all $j = 0, 1, \ldots, N$, i.e.

$$m_j = \int_K x^j \mu(dx), \, \forall\, j = 0, 1, \ldots, N?$$

If such a measure μ does exist we say that μ is a $K-$*representing measure* for m or that m is represented by μ on K. If $N = \infty$ the KMP is said to be *full*, while it is called *truncated* if $N < \infty$. In the following we are going to focus on the full KMP.

Note that there is a bijective correspondence between the set $\mathbb{R}^{\mathbb{N}_0}$ of all sequences of real numbers and the set $(\mathbb{R}[X])^*$ of all linear functionals on $\mathbb{R}[X]$, namely

$$\begin{array}{llllll} \phi: & \mathbb{R}^{\mathbb{N}_0} & \to & (\mathbb{R}[X])^* & & \\ & m := (m_j)_{j\in\mathbb{N}_0} & \mapsto & L_m: & \mathbb{R}[x] & \to \quad \mathbb{R} \\ & & & & p := \sum\limits_j p_j X^j & \mapsto \quad L_m(p) := \sum\limits_j p_j m_j, \end{array}$$

where L_m is called *Riesz' functional*. Indeed

- $\underline{\phi \text{ is injective}}$, because if $m := (m_j)_{j\in\mathbb{N}_0}, m' := (m'_j)_{j\in\mathbb{N}_0} \in \mathbb{R}^{\mathbb{N}_0}$ and $m \neq m'$ then there exists $j \in \mathbb{N}_0$ s.t. $m_j \neq m'_j$, i.e. $L_m(x^j) \neq L_{m'}(x^j)$, and so $\phi(m) = L_m \neq L_{m'} = \phi(m')$.
- $\underline{\phi \text{ is surjective}}$, because for any $\ell \in (\mathbb{R}[X])^*$ the sequence $m := (\ell(X^j))_{j\in\mathbb{N}_0}$ is such that $\phi(m) = \ell$. In fact, for any $p := \sum\limits_j p_j X^j \in \mathbb{R}[X]$ we have

  $$L_m(p) = \sum_j p_j \ell(X^j) = \ell\left(\sum_j p_j X^j\right) = \ell(p) \text{ and, hence, } \phi(m) = L_m = \ell.$$

In virtue of this correspondence, we can always reformulate the full KMP in terms of linear functionals.

Problem 2.1.3 (The one-dimensional $K-$Moment Problem (KMP))**.**
Given a closed subset K of $\mathbb{R}$ and $L : \mathbb{R}[X] \to \mathbb{R}$ linear, does there exists a non-negative Radon measure μ supported in K s.t. $L(p) = \int p d\mu$, $\forall p \in \mathbb{R}[X]$?

If such a measure exists we say that μ is a $K-$*representing measure* for L and that it is a solution to the $K-$moment problem for L.

This reformulation makes clearly how to generalize the statement of the one-dimensional KMP to higher dimensions (see also [15, Section 5.2.2]). Let $n \in \mathbb{N}$ and $\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$.

Problem 2.1.4 (The n-dimensional $K-$Moment Problem (KMP))**.**
Given a closed subset K of $\mathbb{R}^n$ and $L:\mathbb{R}[\underline{X}]\to\mathbb{R}$ linear, does there exists a non-negative Radon measure μ supported in K s.t. $L(p)=\int pd\mu$, $\forall p\in\mathbb{R}[\underline{X}]$?

We can clearly consider also infinite dimensional settings, e.g. by replacing $\mathbb{R}[X_1,\dots,X_n]$ with $\mathbb{R}[X_i : i\in\Omega]$, where Ω is an infinite index set or replacing the polynomial algebra by any infinitely generated unital commutative $\mathbb{R}-$algebra. Let us then give a formulation of the $K-$moment problem general enough to encompass all the above mentioned instances.

Given a unital commutative $\mathbb{R}-$algebra A, recall that we denote by $X(A)$ its character space of A (see Section 1.3.2). We endow the character space $X(A)$ with the weakest topology $\tau_{X(A)}$ on $X(A)$ s.t. all Gelfand transforms are continuous, i.e. $\hat{a}:X(A)\to\mathbb{R}$, $\hat{a}(\alpha):=\alpha(a)$ is continuous for all $a\in A$. A basis for $\tau_{X(A)}$ is given by

$$\mathcal{N}:=\left\{\bigcap_{i=1}^{n}\hat{a_i}^{-1}(U_{a_i}) : a_1,\dots,a_n\in A, U_{a_1},\dots,U_{a_n}\text{open in }\mathbb{R}, n\in\mathbb{N}\right\}.$$

Remark 2.1.5. *$X(A)$ can be seen as a subset of $\mathbb{R}^A$ via the embedding:*

$$\begin{array}{rccl}\pi: & X(A) & \to & \mathbb{R}^A\\ & \alpha & \mapsto & \pi(\alpha):=(\alpha(a))_{a\in A}=(\hat{a}(\alpha))_{a\in A}.\end{array}$$

If we equip $\mathbb{R}^A$ with the product topology τ_{prod}, then $\tau_{X(A)}$ coincides with the topology τ_π induced by π on $X(A)$ from $(\mathbb{R}^A,\tau_{prod})$, i.e.

$$\tau_{X(A)}\equiv\left\{\pi^{-1}(O) : O\in\tau_{prod}\right\}.$$

Hence, π is a topological embedding and the space $\left(X(A),\tau_{X(A)}\right)$ is Hausdorff.

Proof. Let $a\in A$. Then π is $\tau_\pi-$continuous and the projection $p_a:\mathbb{R}^A\to\mathbb{R}$, $p_a((x_b)_{b\in A}):=x_a$ is $\tau_{prod}-$continuous. Hence, $\hat{a}=p_a\circ\pi$ is $\tau_\pi-$continuous and so $\tau_{X(A)}\subseteq\tau_\pi$.

Conversely, let $O\in\tau_{prod}$. Then there exist $n\in\mathbb{N}$, $b_1,\dots,b_n\in A$ and $U_{b_1},\dots,U_{b_n}$ open in $\mathbb{R}$ such that $\prod\limits_{i=1}^{n}U_{b_i}\times\prod\limits_{a\in A\setminus\{b_1,\dots,b_n\}}\mathbb{R}\subseteq O$. Hence, $\pi^{-1}(O)\supseteq\pi^{-1}\left(\bigcap_{i=1}^n p_{b_i}^{-1}(U_{b_i})\right)=\bigcap_{i=1}^n\pi^{-1}\left(p_{b_i}^{-1}(U_{b_i})\right)=\bigcap_{i=1}^n\hat{b_i}^{-1}(U_{b_i})\in\mathcal{N}$ and so $\tau_\pi\subseteq\tau_{X(A)}$ $\square$

We are now ready to introduce the general formulation of KMP announced above.

Problem 2.1.6 (The KMP for unital commutative $\mathbb{R}-$algebras)**.**
Let A be a unital commutative $\mathbb{R}-$algebra. Given a closed subset $K \subseteq X(A)$ and $L : A \to \mathbb{R}$ linear, does there exist a non-negative Radon measure μ on $X(A)$ supported on K and such that $L(a) = \int_{X(A)} \hat{a}(\alpha)\mu(d\alpha), \forall a \in A$?

Note that for $A = \mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \ldots, X_n]$ Problem 2.1.6 reduces to Problem 2.1.4 by means of the correspondence $X(\mathbb{R}[\underline{X}]) \cong \mathbb{R}^n$ introduced in Proposition 1.3.26.

2.2 Riesz-Haviland's Theorem

Let A be a unital commutative $\mathbb{R}-$algebra. Given a subset K of $X(A)$, we denote by

$$\mathrm{Psd}(K) := \{a \in A : \hat{a} \geq 0 \text{ on } K\}.$$

A necessary condition for the existence of a solution to Problem 2.1.6 is clearly that L is nonnegative on $\mathrm{Psd}(K)$. In fact, if there exists a $K-$representing measure μ for L then for all $a \in \mathrm{Psd}(K)$ we have

$$L(a) = \int_{X(A)} \hat{a}(\alpha)\mu(d\alpha) \geq 0$$

since μ is nonnegative and supported on K and $\hat{a}$ is nonnegative on K.

It is then natural to ask if the non-negativity of L on $\mathrm{Psd}(K)$ is also sufficient. For $A = \mathbb{R}[X_1, \ldots, X_n]$ a positive answer is provided by the so-called Riesz-Haviland theorem (see [34, 14]).

Theorem 2.2.1 (Classical Riesz-Haviland Theorem)**.** *Let $K \subseteq \mathbb{R}^n$ closed and $L : \mathbb{R}[X_1, \ldots, X_n] \to \mathbb{R}$ linear. Then L has a $K-$representing measure if and only if $L(\mathrm{Psd}(K)) \subseteq [0, +\infty)$.*

An analogous result also holds in the general setting.

Theorem 2.2.2 (Generalized Riesz-Haviland Theorem)**.** *Let $K \subseteq X(A)$ closed and $L : A \to \mathbb{R}$ linear. Suppose there exists $p \in A$ such that $\hat{p} \geq 0$ on K and for all $n \in \mathbb{N}$ the set $\{\alpha \in K : \hat{p}(\alpha) \leq n\}$ is compact. Then L has a $K-$representing measure if and only if $L(\mathrm{Psd}(K)) \subseteq [0, +\infty)$.*

This theorem reduces the solvability of the $K-$moment problem to the problem of characterizing $\mathrm{Psd}(K)$ establishing the beautiful duality between these two problems.

We will prove both Theorems 2.2.1 and 2.2.2 as corollaries of the following more general result for which we need some notation. Given a topological space (X,τ), we denote by $\mathcal{C}(X)$ the space of all continuous real valued functions defined and by $\mathcal{C}_c(X)$ the subspace of all functions in $\mathcal{C}(X)$ having compact support $\text{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}^{\tau}$.

Theorem 2.2.3. *Let A be a unital commutative $\mathbb{R}-$algebra, χ a Hausdorff space and $\hat{}: A \to \mathcal{C}(\chi)$ a $\mathbb{R}-$algebra homomorphism. Suppose that*

$$\exists\, p \in A \text{ s.t. } \hat{p} \geq 0 \text{ on } \chi \text{ and } \forall j \in \mathbb{N},\ \chi_j := \{\alpha \in \chi : \hat{p}(\alpha) \leq j\} \text{ is compact.} \tag{2.1}$$

If $L : A \to \mathbb{R}$ is linear and s.t. $L(a) \geq 0$ for all $a \in A$ with $\hat{a} \geq 0$ on χ, then there exists a Radon measure μ on χ such that $L(a) = \int \hat{a} d\mu$, for all $a \in A$.

Remark 2.2.4. (2.1) *implies that χ is locally compact, i.e. for any $x \in \chi$ there exists a compact neighbourhood of x.*

Proof.
Let $x \in \chi$ and $j \in \mathbb{N}$ such that $\hat{p}(x) < j$. Then $U := \big\{y \in \chi \mid \hat{p}(y) < j\big\} \subseteq \chi_j$, $x \in U$, and U is open (since $U = \hat{p}^{-1}\big((-\infty, j)\big)$ and $\hat{p} \in \mathcal{C}(\chi)$). Hence, U is an open neighbourhood of x and so $\overline{U}$ is a closed neighbourhood of x contained in χ_j, which is compact. Then, $\overline{U}$ is a compact neighbourhood of x. □

Proof. of Theorem 2.2.1
Let $\chi := K$ be a closed subset of $\mathbb{R}^n$, $A := \mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$, $\hat{}:$ $\mathbb{R}[X_1, \ldots, X_n] \to \mathcal{C}(K)$ defined by $\hat{f} := f \restriction_K$, and $p := \sum_{i=1}^n X_i^2$ i.e. $p = \|\underline{X}\|^2$, where $\|\cdot\|$ is the euclidean norm on $\mathbb{R}^n$. Then $\hat{p} \geq 0$ on K and for any $j \in \mathbb{N}$ the $\chi_j = \{x \in K : \|x\|^2 \leq j\}$ is compact. Hence, (2.1) holds and the conclusion follows by Theorem 2.2.3. □

Proof. of Theorem 2.2.2
Let $\chi := K$ be a closed subset of $X(A)$ endowed with the subset topology induced by $\tau_{X(A)}$ which makes K into a Hausdorff space. Define the map

$$\begin{array}{rcl} \hat{}: A & \to & \mathcal{C}(K) \\ a & \mapsto & \hat{a} \restriction_K, \end{array}$$

where $\hat{a}$ is the Gelfand transform of a. This is well-defined as the Gelfand transform of a restricted to K is a continuous $\mathbb{R}-$algebra homomorphism. Then the conclusion follows by Theorem 2.2.3. □