Theorem 2.2.3 was most probably known since at least the sixties as it can be derived from a theorem due to Choquet in [6]. However, we propose a proof due to Marshall, see [29, Theorem 3.2.2] or [28, Theorem 3.1], and based on the following famous result.

**Theorem 2.2.5** (Riesz-Markov-Kakutani theorem)**.** *Let $\chi$ be a locally compact Hausdorff space. If $L : \mathcal{C}_c(\chi) \to \mathbb{R}$ is a positive linear functional, i.e. $L(f) \geq 0$ for all $f \in \mathcal{C}_c(\chi)$ such that $f \geq 0$ on $\chi$, then there exists a unique Borel regular measure $\mu$ on $\chi$ such that $L(f) = \int f d\mu$ for all $f \in \mathcal{C}_c(\chi)$.*

*Proof.* (see e.g. [20, Theorem 16, p.77]) □

Recall that a Borel regular measure $\mu$ on the Hausdorff space $(\chi, \tau)$ is a measure defined on the Borel $\sigma-$algebra $\mathcal{B}_\tau$ such that $\mu$ is both inner regular and outer regular, where $\mu$ outer regular means that for all $B \in \mathcal{B}_\tau$, $\mu(B) = \inf\{\mu(O) : O \supseteq B \text{open}\}$. Note that a finite Borel regular measure is in particular a Radon measure.

*Proof. of Theorem 2.2.3*
Let $\hat{A} := \{\hat{a} : a \in A\}$ and $\mathcal{B}(\chi) := \{f \in \mathcal{C}(\chi) : \exists a \in A \text{ s.t.} |f| \leq |\hat{a}| \text{ on } \chi\}$. Since $\hat{}: A \to \mathcal{C}(\chi)$ is an $\mathbb{R}-$algebra homomorphism, we have that both $\hat{A}$ and $\mathcal{B}(\chi)$ are subalgebras of $\mathcal{C}(\chi)$ and $\hat{A} \subseteq \mathcal{B}(\chi) \subseteq \mathcal{C}(\chi)$.

**Claim 1**: $\mathcal{C}_c(\chi)$ is a subalgebra of $\mathcal{B}(\chi)$.

*Proof of Claim 1.*
Clearly, $\mathcal{C}_c(\chi)$ equipped with the pointwise operations of addition and multiplication is an $\mathbb{R}-$algebra. Moreover, if $f \in \mathcal{C}_c(\chi)$ then $f$ is bounded above on $\chi$, and so there exists $k \in \mathbb{N}$ s.t. $|f| \leq k$ on $\chi$. Since $k \in A$, we have that $|f| \leq \hat{k}$ on $\chi$, i.e. $f \in \mathcal{B}(\chi)$. Hence, $\mathcal{C}_c(\chi)$ is a subalgebra of $\mathcal{B}(\chi)$. □(Claim 1)

Define $\overline{L} : \hat{A} \to \mathbb{R}$ as $\overline{L}(\hat{a}) = L(a)$ for all $a \in A$.

**Claim 2**: $\overline{L}$ is a well-defined linear functional on $\hat{A}$.

*Proof of Claim 2.*
It is enough to prove that

$$\forall a \in A, \ \hat{a} = 0 \Rightarrow L(a) = 0. \tag{2.2}$$

In fact, (2.2) implies that $\overline{L}(a) = \overline{L}(b)$ for any $a, b \in A$ such that $\hat{a} = \hat{b}$, i.e. $\overline{L}$ is well-defined. Also, using (2.2) together with the assumptions that $\hat{}$ is a

$\mathbb{R}-$algebra homomorphism and $L$ is linear, we obtain that for any $a, b \in A$ and $\lambda \in \mathbb{R}$

$$\overline{L}(\hat{a}+\hat{b}) \overset{(2.2)}{=} \overline{L}(\widehat{a+b}) = L(a+b) = L(a)+L(b) = \overline{L}(\hat{a})+\overline{L}(\hat{b})$$

and

$$\overline{L}(\lambda\hat{a}) \overset{(2.2)}{=} \overline{L}(\widehat{\lambda a}) = L(\lambda a) = \lambda L(a) = \lambda\overline{L}(\hat{a}).$$

Let us then show that (2.2) holds. If $\hat{a}=0$ then $\hat{a} \geq 0$ and $-\hat{a} = \widehat{-a} \geq 0$. These respectively imply that $L(a) \geq 0$ and $L(-a) \geq 0$, which together yield $L(a) = 0$, i.e. $\overline{L}(\hat{a}) = 0$.

□(Claim 2)

**Claim 3**: $\overline{L} : \hat{A} \to \mathbb{R}$ extends to a linear functional $\overline{\overline{L}} : \mathcal{B}(\chi) \to \mathbb{R}$ s.t. $\overline{\overline{L}}(f) \geq 0$ for all $f \in \mathcal{B}(\chi)$ with $f \geq 0$ on $\chi$.

*Proof of Claim 3.*

Consider the collection $\mathcal{P}$ of all pairs $\left(V, \overline{\overline{L}}\right)$, where $V$ is a $\mathbb{R}-$subspace of $\mathcal{B}(\chi)$ containing $\hat{A}$ and $\overline{\overline{L}}$ is an extension of $\overline{L} : \hat{A} \to \mathbb{R}$ such that $\overline{\overline{L}}(f) \geq 0$ for all $f \in V$ with $f \geq 0$ on $\chi$. Define the following partial order on $\mathcal{P}$

$$\left(V_1, \overline{\overline{L}}_1\right) \subseteq \left(V_2, \overline{\overline{L}}_2\right) \Longleftrightarrow V_1 \subseteq V_2 \text{ and } \overline{\overline{L}}_2 \restriction_{V_1} = \overline{\overline{L}}_1.$$

- $\mathcal{P}$ is non-empty since $\left(\hat{A}, \overline{L}\right)$ belongs to it. In fact, for any $a \in A$ s.t. $\hat{a} \geq 0$ on $\chi$ we have $\overline{L}(\hat{a}) = L(a) \geq 0$, where the latter inequality holds by assumption on $L$.
- Every chain in $\mathcal{P}$ has an upper bound. Indeed, for any $\{(V_i, \ell_i), i \in J\}$ chain in $\mathcal{P}$, the pair $\left(\bigcup_{i\in J} V_i, \ell\right)$ is an upper bound, where the functional $\ell : \bigcup_{i\in J} V_i \to \mathbb{R}$ is linear and such that $\ell \restriction_{V_i} = \ell_i$ for all $i \in J$.

Then by Zorn's lemma there exists be a maximal element $\left(B, \overline{\overline{L}}\right)$ in $\mathcal{P}$.

We want to show that $B = \mathcal{B}(\chi)$.

Suppose that this is not the case and let $g \in \mathcal{B}(\chi) \setminus B$. If $f_1, f_2 \in B$ s.t. $f_1 \leq g$ and $g \leq f_2$ on $\chi$, then $f_1 \leq f_2$ on $\chi$, and so $\overline{\overline{L}}(f_1) \leq \overline{\overline{L}}(f_2)$. Therefore,

$$\mathcal{U} := \{\overline{\overline{L}}(f_1) : \; f_1 \in B, f_1 \leq g \text{ on } \chi\} \text{ and } \Theta := \{\overline{\overline{L}}(f_2) : \; f_2 \in B, g \leq f_2 \text{ on } \chi\}$$

are such that $u \leq \theta$ for all $u \in \mathcal{U}$ and $\theta \in \Theta$. Moreover, $\mathcal{U}$ and $\Theta$ are both non-empty. [Indeed, as $g \in \mathcal{B}(\chi)$, there exists $a \in A$ s.t. $|g| \leq |\hat{a}|$ on $\chi$ and so $|\hat{a}| \leq \dfrac{\hat{a}^2+1}{2} \in \hat{A}$ (since $\left(\hat{a} \pm 1\right)^2 \geq 0$), which in turns gives that

$f_1 := -\dfrac{\hat{a}^2+1}{2} \in \hat{A}$ and $f_2 := \dfrac{\hat{a}^2+1}{2} \in \hat{A}$ are such that $f_1 \leq g \leq f_2$.] The completeness of $\mathbb{R}$ ensures that

$$\exists\ e \in \mathbb{R} \text{ s.t. } \sup(\mathcal{U}) \leq e \leq \inf(\Theta). \tag{2.3}$$

We can now linearly extend $\overline{\overline{L}}$ from $B$ to $B + \mathbb{R}g \subseteq \mathcal{B}(\chi)$ by setting $\overline{\overline{L}}(g) := e$ and so $\overline{\overline{L}}(f + dg) := \bar{L}(f) + de$ for all $d \in \mathbb{R}$ and $f \in B$. Then the following holds

$$\forall\ f + dg \in B + \mathbb{R}g,\ f + dg \geq 0 \text{ on } \chi \Rightarrow \overline{\overline{L}}(f + dg) \geq 0, \tag{2.4}$$

which yields $\left(B + \mathbb{R}g, \overline{\overline{L}}\right) \supseteq (B, \overline{\overline{L}})$ and so contradicts the maximality of $(B, \overline{\overline{L}})$, proving that $B = \mathcal{B}(\chi)$. To show that (2.4) holds, we need to distinguish three cases.

Case 1: If $d = 0$ and $f + dg \in B + \mathbb{R}g$ is s.t. $f + dg \geq 0$ on $\chi$, then $\overline{\overline{L}}(f) \geq 0$ since $\left(B, \overline{\overline{L}}\right) \in \mathcal{P}$.

Case 2: If $d > 0$ and $f + dg \in B + \mathbb{R}g$ is s.t. $f + dg \geq 0$ on $\chi$, then $-\frac{f}{d} \leq g$ on $\chi$. Hence, $\overline{\overline{L}}\left(-\frac{f}{d}\right) \in \mathcal{U}$ and so by (2.3) we have $\overline{\overline{L}}\left(-\frac{f}{d}\right) \leq e = \overline{\overline{L}}(g)$, i.e. $0 \leq \overline{\overline{L}}(g) - \overline{\overline{L}}\left(-\frac{f}{d}\right) = \overline{\overline{L}}\left(g + \frac{f}{d}\right) = \frac{1}{d}\overline{\overline{L}}\left(f + gd\right)$. Then $\overline{\overline{L}}\left(f + gd\right) \geq 0$.

Case 3: If $d < 0$ and $f + dg \in B + \mathbb{R}g$ is s.t. $f + dg \geq 0$ on $\chi$, then $-\frac{f}{d} \geq g$ on $\chi$. Hence, $\overline{\overline{L}}\left(-\frac{f}{d}\right) \in \Theta$ and so by (2.3) we have $\overline{\overline{L}}\left(-\frac{f}{d}\right) \geq e = \overline{\overline{L}}(g)$, i.e. $0 \leq \overline{\overline{L}}(g) - \overline{\overline{L}}\left(-\frac{f}{d}\right) = \overline{\overline{L}}\left(g + \frac{f}{d}\right) = -\frac{1}{d}\overline{\overline{L}}\left(f + gd\right)$. Then $\overline{\overline{L}}\left(f + gd\right) \geq 0$.

□(Claim 3)

By Claim 1, we know that $\mathcal{C}_c(\chi) \subseteq \mathcal{B}(\chi)$ and so $\overline{\overline{L}}$ is in particular defined on $\mathcal{C}_c(\chi)$ and such that $\overline{\overline{L}}(f) \geq 0$ for all $f \in \mathcal{C}_c(\chi)$ with $f \geq 0$ on $\chi$. This together with Remark 2.2.4 guarantees that we can apply Theorem 2.2.5 and, hence, that

$$\exists\ \mu \text{ Borel regular measure on } \chi \text{ s.t. } \overline{\overline{L}}(f) = \int f d\mu,\ \ \forall f \in \mathcal{C}_c(\chi). \tag{2.5}$$

**Main Claim**: $\overline{\overline{L}}(f) = \int f d\mu, \forall f \in \mathcal{B}(\chi)$.

*Proof of Main Claim.*

Let $f \in \mathcal{B}(\chi)$. W.l.o.g. we can assume that $f \geq 0$ on $\chi$, since $f = f_+ - f_-$ where $f_+ := \max\{f, 0\}$ and $f_- := -\min\{f, 0\}$. Set $q := f + \hat{p}$ where $p$ is the one in (2.1). Then $q \in \mathcal{B}(\chi)$.

For each $j \in \mathbb{N}$, define $\chi_j^{'} := \{x \in \chi \mid q(x) \leq j\}$. Then

- $\forall\ j \in \mathbb{N}, \chi_j^{'}$ is compact. Indeed, for all $x \in \chi$ we have that $q(x) \geq \hat{p}(x)$ and so that $\chi_j^{'} \subseteq \chi_j$, which yields that $\chi_j^{'}$ is closed subset of a compact set and so itself compact.
- $\chi_j^{'} \subseteq \chi_{j+1}^{'}$ and $\chi = \bigcup_j \chi_j^{'}$ .

Subclaim 1: For each $j \in \mathbb{N}$, there exists $f_j \in \mathcal{C}_c(\chi)$ such that $0 \leq f_j \leq f$, $f_j = f$ on $\chi_j^{'}$ and $f_j = 0$ on $\chi \setminus \chi_{j+1}^{'}$.

*Proof of Subclaim 1.*

For each $j \in \mathbb{N}$, let us set $Y_j' = \{x \in \chi_{j+1}' \mid j + \frac{1}{2} \leq q(x) \leq j+1\}$. Then $Y_j'$ and $\chi_j'$ are disjoint closed subsets of $\chi_{j+1}'$. Applying Urysohn's lemma, we get that there exists $g_j : \chi_{j+1}' \to [0,1]$ continuous such that $g_j = 0$ on $Y_j'$ and $g_j = 1$ on $\chi_j'$. We can extend $g_j$ to the whole $\chi$ by setting $g_j = 0$ on $\chi \setminus \chi_{j+1}'$. Then $f_j := f \cdot g_j$ is such that

- $0 \leq f_j \leq f$ on $\chi$, since $0 \leq g_j \leq 1$ on $\chi$.
- $f_j = f \cdot g_j = f$ on $\chi_j'$, since $g_j = 1$ on $\chi_j'$.
- $f_j = f \cdot g_j = 0$ on $\chi \setminus \chi_{j+1}'$, since $g_j = 0$ on $\chi \setminus \chi_{j+1}'$.

In particular, $\mathrm{supp}(f_j) \subseteq \chi_{j+1}'$ is compact, as closed subset of a compact set, and so $f_j \in \mathcal{C}_c(\chi)$.

$\square$(Subclaim 1)

Then $(f_j)_{j\in\mathbb{N}}$ is a non-decreasing sequence of non-negative functions in $C_c(\chi)$ which pointwise converges to $f$ in $\chi$. Indeed, for all $j \in \mathbb{N}$ and all $x \in \chi$, we easily get from Subclaim 1 that $0 \leq f_j(x) \leq f_{j+1}(x)$ and $\lim_{j\to\infty} f_j(x) = f(x)$. Hence, we can apply the Monotone Convergence Theorem, which ensures that

$$\int f d\mu = \lim_{j\to\infty} \int f_j d\mu \overset{(2.5)}{=} \lim_{j\to\infty} \overline{\overline{L}}(f_j).$$

Hence, the proof of the Main Claim is complete once we show that

Subclaim 2: $\overline{\overline{L}}(f) = \lim\limits_{j\to\infty} \overline{\overline{L}}(f_j)$.

*Proof of Subclaim 2.*

Let $j \in \mathbb{N}$. First of all, let us show that

$$\frac{q^2}{j} \geq f - f_j \geq 0 \text{ on } \chi. \tag{2.6}$$

From Subclaim 1 we know that $f = f_j$ on $\chi_j'$, so clearly $\frac{q^2}{j} \geq f - f_j = 0$ on $\chi_j'$. Moreover, for any $x \in \chi \setminus \chi_j'$, we have $q(x) > j$ and so

$$q^2(x) > jq(x) = j\big(f(x) + \hat{p}(x)\big) \geq jf(x) \geq \big(f(x) - f_j(x)\big),$$

which yields $\dfrac{q^2(x)}{j} \geq (f - f_j)(x)$ for all $x \in \chi$.

Now (2.6) implies that $\overline{\overline{L}}\left(\frac{q^2}{j} - (f - f_j)\right) \geq 0$ and $\overline{\overline{L}}\,(f - f_j) \geq 0$. Hence, $\overline{\overline{L}}\left(\frac{q^2}{j}\right) \geq \overline{\overline{L}}\,(f - f_j) \geq 0$, i.e. $\frac{1}{j}\overline{\overline{L}}\left(q^2\right) \geq \overline{\overline{L}}\,(f - f_j) \geq 0$. Then passing to the limit for $j \to \infty$ we obtain that $\lim\limits_{j\to\infty} \overline{\overline{L}}\,(f - f_j) = 0$ and so $\lim\limits_{j\to\infty} \overline{\overline{L}}(f_j) = \overline{\overline{L}}(f)$.

□(Subclaim 2)

□(Main Claim)

Since $\hat{A} \subseteq \mathcal{B}(\chi)$, the Main Claim implies that for all $a \in A$ we have $\overline{\overline{L}}(\hat{a}) = \int \hat{a} d\mu$. This together with the definition of $\overline{L}$ and Claim 3 gives that

$$L(a) = \overline{L}(\hat{a}) = \overline{\overline{L}}(\hat{a}) = \int \hat{a} d\mu, \forall a \in A, \tag{2.7}$$

which yields the conclusion as $\mu$ is a finite Borel regular measure and so Radon. Indeed, using (2.7), we get that $L(1) = \int \hat{1} d\mu = \mu(\chi)$ and so that $\mu$ is finite.

□(Proof of Theorem 2.2.3)