which yields $\dfrac{q^2(x)}{j} \geq (f-f_j)(x)$ for all $x \in \chi$.

Now (2.6) implies that $\overline{\overline{L}}\left(\frac{q^2}{j}-(f-f_j)\right) \geq 0$ and $\overline{\overline{L}}\,(f-f_j) \geq 0$. Hence, $\overline{\overline{L}}\left(\frac{q^2}{j}\right) \geq \overline{\overline{L}}\,(f-f_j) \geq 0$, i.e. $\frac{1}{j}\overline{\overline{L}}\left(q^2\right) \geq \overline{\overline{L}}\,(f-f_j) \geq 0$. Then passing to the limit for $j \to \infty$ we obtain that $\lim_{j\to\infty} \overline{\overline{L}}\,(f-f_j) = 0$ and so $\lim_{j\to\infty} \overline{\overline{L}}(f_j) = \overline{\overline{L}}(f)$.

□(Subclaim 2)

□(Main Claim)

Since $\hat{A} \subseteq \mathcal{B}(\chi)$, the Main Claim implies that for all $a \in A$ we have $\overline{\overline{L}}(\hat{a}) = \int \hat{a}\,d\mu$. This together with the definition of $\overline{L}$ and Claim 3 gives that

$$L(a) = \overline{L}(\hat{a}) = \overline{\overline{L}}(\hat{a}) = \int \hat{a}\,d\mu, \forall a \in A, \tag{2.7}$$

which yields the conclusion as μ is a finite Borel regular measure and so Radon. Indeed, using (2.7), we get that $L(1) = \int \hat{1}\,d\mu = \mu(\chi)$ and so that μ is finite.

□(Proof of Theorem 2.2.3)

2.3 Solving the KMP through characterizations of $\mathrm{Psd}(K)$

The Riesz-Haviland theorem 2.2.1 establishes a beautiful duality between the $K-$moment problem and the problem of characterizing $\mathrm{Psd}(K)$. Hence, thanks to this result we can obtain necessary and sufficient conditions to solve the KMP using the characterizations of $\mathrm{Psd}(K)$ introduced in the previous chapter. For example, combining Riesz-Haviland's theorem with Theorem 1.3.9 about saturation of preorderings we obtain the following.

Corollary 2.3.1. *Let $L : \mathbb{R}[X] \to \mathbb{R}$ be linear and K a non-empty bcsas of $\mathbb{R}$ with natural description $S_{nat} = \{g_1, \ldots, g_s\}$. Then there exists a $K-$representing measure for L if and only if $L(h^2 g_1^{e_1} \ldots g_s^{e_s}) \geq 0$ for all $h \in \mathbb{R}[X]$ and all $e_1, \ldots, e_s \in \{0,1\}$.*

Proof.
By Theorem 2.2.1, the existence of a $K-$representing measure for L is equivalent to the non-negativity of L on $\mathrm{Psd}(K)$. The latter is in turn equivalent to the non-negativity of L on the preordering $T_{S_{nat}}$ associated to the natural description S_{nat} of K, since Theorem 1.3.9 ensures that $\mathrm{Psd}(K) = T_{S_{nat}}$. Hence, the conclusion directly follows from the linearity of L as

$$T_{S_{nat}} = \left\{ \sum_{e=(e_1,\ldots,e_s)\in\{0,1\}^s} \sigma_e\, g_1^{e_1} \ldots g_s^{e_s} : \sigma_e \in \sum \mathbb{R}[X]^2, e \in \{0,1\}^s \right\}.$$

□

Corollary 2.3.1 allows to derive the most classical results about the one-dimensional KMP. Indeed, we have the following

- If $K = \mathbb{R}$, then $S_{nat} = \{\emptyset\}$ and so Corollary 2.3.1 becomes
  Theorem 2.3.2 (Hamburger 1921)**.**
  A linear functional $L : \mathbb{R}[X] \to \mathbb{R}$ has a $\mathbb{R}-$representing measure if and only if $L(h^2) \geq 0$ for all $h \in \mathbb{R}[X]$.
- If $K = [0, +\infty)$, then $S_{nat} = \{X\}$ and so Corollary 2.3.1 becomes
  Theorem 2.3.3 (Stietjes 1885)**.**
  A linear functional $L : \mathbb{R}[X] \to \mathbb{R}$ has a $\mathbb{R}^+-$representing measure if and only if $L(h^2) \geq 0$ and $L(Xh^2) \geq 0$ for all $h \in \mathbb{R}[X]$.
- If $K = [0, 1]$, then $S_{nat} = \{X, 1 - X\}$. Hence, using Corollary 2.3.1 together with the observation that $X(1-X) = X(1-X)^2 + (1-X)X^2$, we obtain
  Theorem 2.3.4 (Hausdorff 1923)**.**
  A linear functional $L : \mathbb{R}[X] \to \mathbb{R}$ has a $[0, 1]-$representing measure if and only if $L(h^2) \geq 0$, $L(Xh^2) \geq 0$ and $L((1 - X)h^2) \geq 0$ for all $h \in \mathbb{R}[X]$.

These classical results were obtained without using Riesz-Haviland theorem, but through methods involving the analysis of the so-called *Hankel matrix* or *moment matrix* associated to the starting functional. In fact, we will see that any condition of the form $L(gh^2) \geq 0$ for all $h \in \mathbb{R}[X]$ and some $g \in \mathbb{R}[X]$ can be translated into the positive semidefiniteness of a certain matrix obtained from the putative moment sequence $(L(X^j))_{j\in\mathbb{N}_0}$.

Let us introduce these concepts for any dimension $n \in \mathbb{N}$.

Definition 2.3.5. *A sequence $m := (m_\alpha)_{\alpha\in\mathbb{N}_0^n}$ of real numbers is called* positive semidefinite (psd) *if*

$$\sum_{\alpha,\beta\in F} c_\alpha c_\beta m_{\alpha+\beta} \geq 0, \quad \forall\ F \subset \mathbb{N}_0^n,\ c_\alpha,\ c_\beta \in \mathbb{R}.$$

Definition 2.3.6. *Given a polynomial $g := \sum_{\gamma\in\mathbb{N}_0^n} g_\gamma \underline{X}^\gamma \in \mathbb{R}[X_1, \ldots, X_n]$ and a sequence $m := (m_\alpha)_{\alpha\in\mathbb{N}_0^n}$ of real numbers, we define $g(E)m := \big((g(E)m)_\alpha\big)_{\alpha\in\mathbb{N}_0^n}$, where*

$$(g(E)m)_\alpha := \sum_{\gamma\in\mathbb{N}_0^n} g_\gamma m_{\alpha+\gamma}.$$

Examples 2.3.7.

1. *For $m := (m_j)_{j\in\mathbb{N}_0} = (m_0, m_1, m_2, \ldots)$, $g := X$ and $h := X^3 - 1$ we get:*
   $g(E)m = (m_{j+1})_{j\in\mathbb{N}_0} = (m_1, m_2, m_3, \ldots)$ and
   $h(E)m = (m_{j+3} - 1)_{j\in\mathbb{N}_0} = (m_3 - 1, m_4 - 1, m_5 - 1, \ldots)$.

2. *For* $m := \left(m_{(\alpha_1,\alpha_2)}\right)_{(\alpha_1,\alpha_2)\in\mathbb{N}_0^2}$ *and* $g := 5 - X_1^2 - X_2^2$*, we have that* $(g(E)m)_{(\alpha_1,\alpha_2)} = 5m_{(\alpha_1,\alpha_2)} - m_{(\alpha_1+2,\alpha_2)} - m_{(\alpha_1,\alpha_2+2)}$.
For instance, $(g(E)m)_{(0,1)} = 5m_{(0,1)} - m_{(2,1)} - m_{(0,3)}$.

Lemma 2.3.8.
Given $L : \mathbb{R}[X_1,\dots,X_n] \to \mathbb{R}$ *linear and* $g := \sum_{\gamma\in\mathbb{N}_0^n} g_\gamma \underline{X}^\gamma \in \mathbb{R}[X_1,\dots,X_n]$*, we have that* $L(gh^2) \geq 0, \forall h \in \mathbb{R}[X_1,\dots,X_n]$ *if and only if* $g(E)m$ *is psd, where* $m := (L(\underline{X}^\alpha))_{\alpha\in\mathbb{N}_0^n}$.

Proof.
For any $\alpha \in \mathbb{N}_0^n$, we have

$$L(g\underline{X}^\alpha) = L\left(\sum_{\gamma\in\mathbb{N}_0^n} g_\gamma \underline{X}^{\gamma+\alpha}\right) = \sum_{\gamma\in\mathbb{N}_0^n} g_\gamma L(\underline{X}^{\gamma+\alpha}) = \sum_{\gamma\in\mathbb{N}_0^n} g_\gamma m_{\gamma+\alpha} = (g(E)m)_\alpha \,.$$

Let $h = \sum_{\beta\in\mathbb{N}_0^n} h_\beta \underline{X}^\beta \in \mathbb{R}[\underline{X}]$. Then $h^2 = \sum_{\beta,\gamma\in\mathbb{N}_0^n} h_\beta h_\gamma \underline{X}^{\beta+\gamma}$ and so

$$\begin{aligned} L(gh^2) &= L\left(g \sum_{\beta,\gamma\in\mathbb{N}_0^n} h_\beta h_\gamma \underline{X}^{\beta+\gamma}\right) \\ &= \sum_{\beta,\gamma\in\mathbb{N}_0^n} h_\beta h_\gamma L(g\underline{X}^{\beta+\gamma}) \\ &= \sum_{\beta,\gamma\in\mathbb{N}_0^n} h_\beta h_\gamma \left(g(E)m\right)_{\beta+\gamma} . \end{aligned}$$

Hence, $L(gh^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$ iff $\sum_{\beta,\gamma\in\mathbb{N}_0^n} h_\beta h_\gamma \left(g(E)m\right)_{\beta+\gamma} \geq 0$ for all $h_\beta, h_\gamma \in \mathbb{R}$, which is equivalent the psd-ness of $g(E)m$. □

Definition 2.3.9. *Let* $L : \mathbb{R}[\underline{X}] \longrightarrow \mathbb{R}$ *be linear and* $g \in \mathbb{R}[\underline{X}]$*. We define the associated symmetric bilinear form as*

$$\begin{aligned} \langle\ ,\ \rangle_g :\ \ \mathbb{R}[\underline{X}] \times \mathbb{R}[\underline{X}] &\to \mathbb{R} \\ (\ p\ ,\ q\ ) &\mapsto \langle p,q\rangle_g := L(pqg) \end{aligned}$$

The moment matrix *associated to* L *and* localized *at* g *is defined to be the infinite real symmetric matrix* $M^g := \left(\langle \underline{X}^\alpha, \underline{X}^\beta\rangle_g\right)_{\alpha,\beta\in\mathbb{N}_0^n} = \left(L(\underline{X}^{\alpha+\beta}\ g)\right)_{\alpha,\beta\in\mathbb{N}_0^n}$*. For* $g = 1$*,* M^1 *is just said the moment matrix associated to* L*.*

Examples 2.3.10.

a) Let $n = 1$, $L : \mathbb{R}[X] \to \mathbb{R}$ linear and set $m := (m_j)_{j \in \mathbb{N}_0}$ with $m_j := L(X^j)$. Then the associated moment matrix is

$$M^1 = \begin{pmatrix} m_0 & m_1 & m_2 & \dots \\ m_1 & m_2 & \iddots & \dots \\ m_2 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} L(1) & L(X) & L(X^2) & \dots \\ L(X) & L(X^2) & \ddots & \dots \\ L(X^2) & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

If $g := X$ then the corresponding localized moment matrix is given by

$$M^g = \begin{pmatrix} m_1 & m_2 & m_3 & \dots \\ m_2 & m_3 & \iddots & \ddots \\ m_3 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} L(X) & L(X^2) & L(X^3) & \dots \\ L(X^2) & L(X^3) & \ddots & \ddots \\ L(X^3) & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

b) Let $n = 2$, $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ linear and set $m := (m_\alpha)_{\alpha \in \mathbb{N}_0^2}$ with $m_{(\alpha_1, \alpha_2)} := L(X_1^{\alpha_1} X_2^{\alpha_2})$. Then the associated moment matrix is

$$\begin{aligned} M^1 &= \begin{pmatrix} m_{00} & m_{10} & m_{01} & m_{20} & m_{11} & \dots \\ m_{10} & m_{20} & m_{11} & m_{30} & \ddots & \ddots \\ m_{01} & m_{11} & m_{20} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \\ &= \begin{pmatrix} L(1) & L(X_1) & L(X_2) & L(X_1^2) & L(X_1X_2) & \dots \\ L(X_1) & L(X_1^2) & L(X_1X_2) & L(X_1^3) & \ddots & \ddots \\ L(X_2) & L(X_1X_2) & L(X_1^3) & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \end{aligned}$$

and if $g = X_1X_2$ then the corresponding localized moment matrix is

$$M^g = \begin{pmatrix} m_{11} & m_{21} & m_{12} & m_{31} & m_{22} & \dots \\ m_{21} & m_{31} & m_{22} & m_{41} & \ddots & \ddots \\ m_{12} & m_{22} & m_{31} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Recall that

Definition 2.3.11. *A real symmetric $N \times N$ matrix A is* positive semidefinite (psd) *if $\underline{y}^t A \underline{y} \geq 0 \ \forall \ \underline{y} \in \mathbb{R}^N$. An infinite real symmetric matrix A is* psd *if $\underline{y}^t A_N \underline{y} \geq 0 \ \forall \ \underline{y} \in \mathbb{R}^N$ and $\forall \ N \in \mathbb{N}$, where A_N is the upper left corner submatrix of order N of A.*

Proposition 2.3.12. *Let $L : \mathbb{R}[\underline{X}] \longrightarrow \mathbb{R}$ be linear and $g \in \mathbb{R}[\underline{X}]$. Then the following are equivalent:*
1) $L(\sigma g) \geq 0 \ \forall \ \sigma \in \sum \mathbb{R}[\underline{X}]^2$.
2) $L(h^2 g) \geq 0 \ \forall \ h \in \mathbb{R}[\underline{X}]$.
3) $\langle \ , \ \rangle_g$ is psd.
4) M^g is psd.
5) $g(E)m$ is psd where $m := (L(\underline{X}^\alpha))_{\alpha \in \mathbb{N}_0^n}$.

Proof.
1) $\Leftrightarrow$ 2) since for any $\sigma \in \sum \mathbb{R}[\underline{X}]^2$, there exist $h_i \in \mathbb{R}[\underline{X}]$ such that $\sigma = \sum_i h_i^2$ and so $L(\sigma g) = \sum_i L(h_i^2 g)$.

2) $\Leftrightarrow$ 3) as $L(h^2 g) = \langle h, h \rangle_g$

3) $\Leftrightarrow$ 4) Indeed, for any $h = \sum_{\gamma \in F} h_\gamma \underline{X}^\gamma \in \mathbb{R}[\underline{X}]$ with $F \subset \mathbb{N}_0^n$ finite, we have

$$\begin{aligned} \langle h, h \rangle_g &= L(\sum_{\beta,\gamma \in F} h_\beta h_\gamma \underline{X}^{\beta+\gamma} g) = \sum_{\beta,\gamma \in F} h_\beta h_\gamma L(g \underline{X}^{\beta+\gamma}) \\ &= \sum_{\beta,\gamma \in F} h_\beta h_\gamma M^g(\beta,\gamma) = y^t M^g_{|F|} y, \end{aligned}$$

where $y := (h_\gamma)_{\gamma \in F}$.

4) $\Leftrightarrow$ 5)
$g(E)m$ is psd iff $\sum_{\beta,\gamma \in \mathbb{N}_0^n} h_\beta h_\gamma \left(g(E)m\right)_{\beta+\gamma} \geq 0$ for all $h_\beta, h_\gamma \in \mathbb{R}$, which is equivalent to the psd-ness of M^g since $\left(g(E)m\right)_{\beta+\gamma} = M^g(\beta,\gamma)$.

5) $\Leftrightarrow$ 1) by Lemma 2.3.8. □

We can then express the Hambuger, Stieltjes and Hausdorff solutions to the KMP in terms of moment matrices.

Theorem 2.3.13.
Given $m := (m_j)_{j\in\mathbb{N}_0}$, the following are equivalent:
a) m is a Hamburger's moment sequence, i.e. has a $\mathbb{R}-$representing measure
b) m is psd
c) M^1 is psd
d) $L_m(h^2) \geq 0$ for all $h \in \mathbb{R}[X]$.

Theorem 2.3.14.
Given $m := (m_j)_{j\in\mathbb{N}_0}$, the following are equivalent:
a) m is a Stieltjes's moment sequence, i.e. has a $\mathbb{R}^+-$representing measure
b) m and $g(E)m$ are both psd
c) M^1 and M^g are both psd
d) $L_m(h^2) \geq 0$ and $L_m(gh^2) \geq 0$ for all $h \in \mathbb{R}[X]$.
where $g := X$.

Theorem 2.3.15.
Given $m := (m_j)_{j\in\mathbb{N}_0}$, the following are equivalent:
a) m is a Hausdorff's moment sequence, i.e. has a $[0, 1]-$representing measure
b) m, $g_1(E)m$ and $g_2(E)m$ are all psd
c) M^1, M^{g_1} and M^{g_2} are all psd
d) $L(h^2) \geq 0$, $L(g_1h^2) \geq 0$ and $L(g_2h^2) \geq 0$ for all $h \in \mathbb{R}[X]$.
where $g_1 := X$ and $g_2 := 1 - X$.

Let us now relate to the KMP the Nichtnegativstellensätze and the closure results introduced in the previous chapter.

Proposition 2.3.16.
Let τ be a locally convex topology on $\mathbb{R}[\underline{X}]$. Given a convex cone C of $\mathbb{R}[\underline{X}]$ and a closed subset K of $\mathbb{R}^n$, the following are equivalent
a) $\mathrm{Psd}(K) \subseteq C_\tau^{\vee\vee}$
b) $\forall\ L \in C_\tau^\vee$, $\exists\ \mu$ $K-$representing measure for L,
where:
$C_\tau^\vee := \{\ell : \mathbb{R}[\underline{X}] \to \mathbb{R}$ *linear* $|\ell$ *is* $\tau-$ *continuous and* $\ell(C) \geq 0\}$ *and*
$C_\tau^{\vee\vee} := \{p \in \mathbb{R}[\underline{X}]\,|\forall\,\ell \in C_\tau^\vee,\ \ell(p) \geq 0\}$.

Proof.
a) $\Rightarrow$ *b)* Let $L \in C_\tau^\vee$, i.e. L is $\tau-$ continuous and non-negative on C. Then $L\left(\overline{C}^\tau\right) \subseteq [0, +\infty)$ and so, by Corollary 1.3.35, $L\left(C_\tau^{\vee\vee}\right) \subseteq [0, +\infty)$. This implies by a) that $L(\mathrm{Psd}(K)) \subseteq [0, +\infty)$ which is equivalenty by Riesz-Haviland Theorem 2.2.1 to the existence of a $K-$representing measure for L.

b) $\Rightarrow$ *a)* By b), we have that $\forall\ L \in C_\tau^\vee$, $L(\mathrm{Psd}(K)) \subseteq [0, +\infty)$, i.e. $L \in (\mathrm{Psd}(K))_\tau^\vee$. Then $C_\tau^\vee \subseteq (\mathrm{Psd}(K))_\tau^\vee$ and so

$$C_\tau^{\vee\vee} \supseteq (\mathrm{Psd}(K))_\tau^{\vee\vee} \supseteq \mathrm{Psd}(K).$$

□

By combining the previous result with Corollary 1.3.42 (respectively 1.3.41 and 1.3.40) and recalling that every linear functional is continuous w.r.t. the finest locally convex topology, we obtain the following results for the KMP.

Corollary 2.3.17. *Let $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ linear and $S := \{g_1, \dots, g_s\} \subset \mathbb{R}[\underline{X}]$ such that the associated bcsas K_S is compact. Then there exists a K_S-representing measure for L if and only if $L(h^2 g_1^{e_1} \cdots g_s^{e_s}) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$, $e_1, \dots, e_s \in \{0, 1\}$.*

Corollary 2.3.18. *Let $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ linear and $S := \{g_1, \dots, g_s\} \subset \mathbb{R}[\underline{X}]$ such that the quadratic module M_S generated by S is Archimedean. Then there exists a K_S-representing measure for L if and only if $L(h^2 g_i) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$ and $i \in \{1, \dots, s\}$*

Corollary 2.3.19. *Let $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ linear and M be an Archimedean $2d-$power module of $\mathbb{R}[\underline{X}]$ with $d \in \mathbb{N}$. Then $\exists$ a K_M-representing measure for L if and only if $L(M) \subseteq [0, +\infty)$.*