By combining the previous result with Corollary 1.3.42 (respectively 1.3.41 and 1.3.40) and recalling that every linear functional is continuous w.r.t. the finest locally convex topology, we obtain the following results for the KMP.

Corollary 2.3.17. *Let $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ linear and $S := \{g_1, \ldots, g_s\} \subset \mathbb{R}[\underline{X}]$ such that the associated bcsas K_S is compact. Then there exists a K_S-representing measure for L if and only if $L(h^2 g_1^{e_1} \cdots g_s^{e_s}) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$, $e_1, \ldots, e_s \in \{0, 1\}$.*

Corollary 2.3.18. *Let $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ linear and $S := \{g_1, \ldots, g_s\} \subset \mathbb{R}[\underline{X}]$ such that the quadratic module M_S generated by S is Archimedean. Then there exists a K_S-representing measure for L if and only if $L(h^2 g_i) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$ and $i \in \{0, 1, \ldots, s\}$, where $g_0 := 1$.*

Corollary 2.3.19. *Let $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ linear and M be an Archimedean $2d-$power module of $\mathbb{R}[\underline{X}]$ with $d \in \mathbb{N}$. Then $\exists$ a K_M-representing measure for L if and only if $L(M) \subseteq [0, +\infty)$.*

Remark 2.3.20. *Corollary 2.3.17 is actually the dual facet of Corollary 1.3.42, since we can also deduce Corollary 1.3.42 from Corollary 2.3.17. Indeed, by Proposition 2.3.16, Corollary 2.3.17 is equivalent to $\mathrm{Psd}(K_S) \subseteq (T_S)^{\vee\vee}_{\tau}$. This together with Corollary 1.3.35 and the fact that $\mathrm{Psd}(K_S) = \bigcap_{x \in K_S} e_x^{-1}([0, +\infty))$ (where $e_x(p) := p(x)$ for all $p \in \mathbb{R}[\underline{X}]$) yields that*

$$\mathrm{Psd}(K_S) \subseteq (T_S)^{\vee\vee}_{\varphi} = \overline{T_S}^{\varphi} \subseteq \overline{\mathrm{Psd}(K_S)}^{\varphi} = \mathrm{Psd}(K_S).$$

Hence, $\mathrm{Psd}(K_S) = \overline{T_S}^{\varphi}$, i.e. Corollary 1.3.42 holds.

A similar argument shows that Corollary 1.3.41 (respectively, Corollary 1.3.40) can be derived from Corollary 2.3.18 (respectively, Corollary 2.3.19).

Proposition 2.3.16 can be easily generalized to any unital commutative $\mathbb{R}-$algebra with the only additional assumption that

$$\exists\, p \in A, \text{s.t. } \hat{p} \geq 0 \text{ on } K \text{ and } \forall\, n \in \mathbb{N},\ \{\alpha \in K : \hat{p}(\alpha) \leq n\} \text{ is compact.} \tag{2.8}$$

This hypothesis is fundamental for the application of the generalized Riesz-Haviland Theorem 2.2.2 and so to get the following.

Proposition 2.3.21. *Let A be a unital commutative $\mathbb{R}-$algebra and C a convex cone of A. Given a locally convex topology τ on A and a closed subset K of $X(A)$ s.t. (2.8) holds, the following are equivalent*
a) $\mathrm{Psd}(K) \subseteq C^{\vee\vee}_{\tau}$
b) $\forall\ L \in C^{\vee}_{\tau}$, $\exists\ \mu$ $K-$representing measure for L,
where:
$C^{\vee}_{\tau} := \{\ell : A \to \mathbb{R}$ *linear* $|\ell$ *is* $\tau -$ *continuous and* $\ell(C) \geq 0\}$ *and*
$C^{\vee\vee}_{\tau} := \{a \in A\, |\forall\, \ell \in C^{\vee}_{\tau},\, \ell(a) \geq 0\}$.

Proof.

a) $\Rightarrow$ *b)* Let $L \in C_\tau^\vee$, i.e. L is $\tau -$ continuous and non-negative on C. Then $L\left(\overline{C}^\tau\right) \subseteq [0, +\infty)$ and so, by Corollary 1.3.35, $L\left(C_\tau^{\vee\vee}\right) \subseteq [0, +\infty)$. This implies by a) that $L(\text{Psd}(K)) \subseteq [0, +\infty)$ which is equivalent by generalized Riesz-Haviland Theorem 2.2.2 to the existence of a $K-$representing measure for L. Note that we can apply the generalized Riesz-Haviland Theorem 2.2.2 since we assumed that (2.8) holds.

b) $\Rightarrow$ *a)* By b), we have that for any $L \in C_\tau^\vee$ there exists a non-negative Radon measure μ supported in K and such that $L(a) = \int \hat{a} d\mu$. Hence, for all $a \in \text{Psd}(K)$ we have $L(a) \geq 0$, i.e. $L \in (\text{Psd}(K))_\tau^\vee$. Then $C_\tau^\vee \subseteq (\text{Psd}(K))_\tau^\vee$ and so $C_\tau^{\vee\vee} \supseteq (\text{Psd}(K))_\tau^{\vee\vee} \supseteq \text{Psd}(K)$. □

By combining Proposition 2.3.21 with Theorem 1.3.45 we get the following result for Problem 2.1.6.

Theorem 2.3.22. *Let (A, ρ) be a unital commutative seminormed $\mathbb{R}-$algebra, $L : A \to \mathbb{R}$ linear, $d \in \mathbb{N}$ and M a $2d-$power module of A. Then there exists a $(K_M \cap \mathfrak{sp}(\rho))-$representing measure for L if and only if L is $\rho-$continuous and $L(M) \subseteq [0, \infty)$.*

Before proving it, let us recall that the Gelfand spectrum $\mathfrak{sp}(\rho)$ is the collection of all $\rho-$continuous characters of A and let us show the following property.

Lemma 2.3.23. *If (A, ρ) is a unital commutative seminormed $\mathbb{R}-$algebra, then the Gelfand spectrum $\mathfrak{sp}(\rho)$ is compact.*

Proof. By Lemma 2.3.8, we know that

$$\begin{aligned} \mathfrak{sp}(\rho) &= \{\alpha \in X(A) : |\alpha(a)| \leq \rho(a), \ \forall\, a \in A\} \\ &= \left\{\alpha \in X(A) : (\hat{a}(\alpha))_{a\in A} \in \prod_{a \in A} [-\rho(a), \rho(a)]\right\}. \end{aligned}$$

Hence, using the embedding

$$\begin{aligned} \pi : \quad X(A) &\to \mathbb{R}^A \\ \alpha &\mapsto \pi(\alpha) := (\alpha(a))_{a\in A} = (\hat{a}(\alpha))_{a\in A}\,. \end{aligned}$$

we have that

$$\pi(\mathfrak{sp}(\rho)) = \pi(X(A)) \cap \prod_{a\in A} [-\rho(a), \rho(a)] \tag{2.9}$$

Since $\pi(X(A))$ is closed in $(\mathbb{R}^A, \tau_{prod})$ (see Sheet 5) and $\prod_{a\in A}([-\rho(a), \rho(a)])$ is compact in $(\mathbb{R}^A, \tau_{prod})$ by Tychonoff theorem, (2.9) ensures that $\pi(\mathfrak{sp}(\rho))$ is a closed subset of a compact set and so it is compact itself.

Let $(U_i)_{i\in I}$ s.t. $U_i \in \tau_{X(A)}$ and $\mathfrak{sp}(\rho) \subseteq \bigcup_{i\in I} U_i$. Then by Remark 2.1.5 for each $i \in I$ there exists $O_i \in \tau_{prod}$ s.t. $\pi^{-1}(O_i) = U_i$. Hence,

$$\mathfrak{sp}(\rho) \subseteq \bigcup_{i\in I} \pi^{-1}(O_i) = \pi^{-1}\left(\bigcup_{i\in I} O_i\right),$$

which implies $\pi(\mathfrak{sp}(\rho)) \subseteq \pi\left(\pi^{-1}\left(\bigcup_{i\in I} O_i\right)\right) \subseteq \bigcup_{i\in I} O_i$. Then the compactness of $\pi(\mathfrak{sp}(\rho))$ guarantees that there exists $J \subset I$ finite and such that $\pi(\mathfrak{sp}(\rho)) \subseteq \bigcup_{i\in J} O_i$, which gives

$$\mathfrak{sp}(\rho) \subseteq \pi^{-1}\left(\pi(\mathfrak{sp}(\rho))\right) \subseteq \pi^{-1}\left(\bigcup_{i\in J} O_i\right) = \bigcup_{i\in J} \pi^{-1}(O_i) = \bigcup_{i\in J} U_i.$$

Hence, $\mathfrak{sp}(\rho)$ is compact. □

Proof. of Theorem 2.3.22
Since (A, ρ) is a seminormed algebra (and so in particular a locally convex t.v.s.) we can apply both Theorem 1.3.45 and Corollary 1.3.35, which yield

$$\mathrm{Psd}(K_M \cap \mathfrak{sp}(\rho)) = \overline{M}^{\rho} = M_{\rho}^{\vee\vee}.$$

Moreover, (2.8) holds by taking $p = 1$. Indeed, $\hat{1} = 1 > 0$ on $X(A)$ and for all $n \in \mathbb{N}$ the set $\{\alpha \in K_M \cap \mathfrak{sp}(\rho) : \hat{1}(\alpha) \leq n\}$ is nothing but $K_M \cap \mathfrak{sp}(\rho)$ which is compact by Lemma 2.3.23.

Suppose that L is $\rho-$continuous and $L(M) \subseteq [0, \infty)$, i.e. $L \in M_{\rho}^{\vee}$. Then Proposition 2.3.21 ensures that there exists a $(K_M \cap \mathfrak{sp}(\rho))-$representing measure for L.

Conversely, suppose that there exists a $(K_M \cap \mathfrak{sp}(\rho))-$representing measure for L. Then clearly $L(M) \subseteq [0, +\infty)$ and for any $a \in A$ we have that

$$|L(a)| \leq \int_{K_M \cap \mathfrak{sp}(\rho)} |\hat{a}(\alpha)|\, d\mu(\alpha) \leq \rho(a)L(1),$$

i.e. L is $\rho-$continuous. □

Remark 2.3.24. *Theorem 2.3.22 is actually the dual facet of Theorem 1.3.45, since we can also deduce Theorem 1.3.45 from Theorem 2.3.22. Indeed, we have already observed that (2.8) holds because of the compactness of* $K_M \cap \mathfrak{sp}(\rho)$

and so we can apply Proposition 2.3.21, which ensures that Theorem 2.3.22 is equivalent to $\text{Psd}(K_M \cap \mathfrak{sp}(\rho)) \subseteq M_\rho^{\vee\vee}$. *This together with Corollary 1.3.35 and the fact that* $\text{Psd}(K_M \cap \mathfrak{sp}(\rho)) = \bigcap_{\alpha \in K_M \cap \mathfrak{sp}(\rho)} \alpha^{-1}([0, +\infty))$ *yields that*

$$\text{Psd}(K_M \cap \mathfrak{sp}(\rho)) \subseteq M_\rho^{\vee\vee} = \overline{M}^\rho \subseteq \overline{\text{Psd}(K_M \cap \mathfrak{sp}(\rho))}^\rho = \text{Psd}(K_M \cap \mathfrak{sp}(\rho)).$$

Hence, $\text{Psd}(K_M \cap \mathfrak{sp}(\rho)) = \overline{M}^\rho$, *i.e. Theorem 1.3.45 holds.*

Theorem 2.3.22 easily extends to the case when A is an arbitrary lmc algebra (i.e. a topological algebra, where the the topology is generated by a family of submultiplicative seminorms).

Theorem 2.3.25. *Let* (A, τ) *be a unital commutative lmc* $\mathbb{R}-$*algebra,* $d \in \mathbb{N}$, M *a* $2d-$*power module of* A *and* $L : A \to \mathbb{R}$ *linear. Then* L *is* $\tau-$*continuous and* $L(M) \subseteq [0, \infty)$ *if and only if there exists a* $(K_M \cap \mathfrak{sp}(\rho))-$*representing measure for* L *for some* $\rho \in \mathcal{F}$, *where* $\mathcal{F}$ *is a directed family of submultiplicative seminorms generating* τ.

Proof. Since (A, τ) is an lmc algebra, there always exists a directed family $\mathcal{F}$ of submultiplicative seminorms generating τ (see [15, Theorem 4.2.14]). Then the $\tau-$continuity of L is equivalent to the $\rho-$continuity of L for some $\rho \in \mathcal{F}$ by [15, Proposition 4.6.1]. Hence, Theorem 2.3.22 guarantees that there exists a $(K_M \cap \mathfrak{sp}(\rho))-$representing measure for L. □

In Theorems 2.3.22 and 2.3.25 as well as in Corollaries 2.3.17, 2.3.18 and 2.3.19 the representing measure are always compactly supported. This gives in turn the uniqueness of the representing measure in each of these results.

Theorem 2.3.26. *If* μ *is a Radon measure on* $X(A)$ *supported on a compact subset* K, *then it is determinate, i.e. any other Radon measure* ν *on* $X(A)$ *such that* $\int \hat{a} d\mu = \int \hat{a} d\nu$ *for all* $a \in A$ *coincides with* μ.

To prove this result we will make use of the Stone-Weirstrass Theorem, which we state here for the convenience of the reader.

Theorem 2.3.27 (Stone-Weirstrass' Theorem)**.** *Let* χ *be a Hausdorff compact topological space and* $\mathcal{C}$ *a subalgebra of* $\mathcal{C}(\chi)$ *containing a non-zero constant function. Then* $\mathcal{C}$ *is dense in* $\mathcal{C}(\chi)$ *if and only if* $\mathcal{C}$ *separates the points of* χ, *i.e. for any* $x \neq y$ *in* χ *there exists* $f \in \mathcal{C}$ *such that* $f(x) \neq f(y)$.

Proof. of Theorem 2.3.26
Let us first show that ν is also supported in K and then that ν coincides with μ.

Suppose that ν is not supported in K. Then there exists $Z \subseteq X(A) \setminus K$ compact and such that $\nu(Z) > 0$. Let $\varepsilon > 0$ such that $\varepsilon < \frac{\nu(Z)}{\mu(K)+\nu(Z)}$. Now $\{\hat{a} : a \in A\}$ is a subalgebra of $\mathcal{C}(X(A))$ which separates the points of $X(A)$, since for any $\alpha_1 \neq \alpha_2$ in $X(A)$ there exists $a \in A$ such that $\alpha_1(a) \neq \alpha_2(a)$, i.e. $\hat{a}(\alpha_1) \neq \hat{a}(\alpha_2)$. Hence, $\{\hat{a} : a \in A\}$ in particular separates the points of $K \cup Z$. Since K and Z are both compact and disjoint, we can apply Urysohn's lemma, which ensures that there exists $g \in \mathcal{C}(K \cup Z)$ such that $g \restriction_K = 0$ and $g \restriction_Z = 1$. Therefore, by Stone-Weirstrass' Theorem 2.3.27 applied to $K \cup Z$, we obtain that there exists $a \in A$ such that $|\hat{a}(\alpha) - g(\alpha)| \leq \varepsilon, \ \forall \ \alpha \in K \cup Z$, i.e.

$$\exists \ a \in A : |\hat{a}(\alpha)| \leq \varepsilon, \ \forall \ \alpha \in K \text{ and } |\hat{a}(\alpha) - 1| \leq \varepsilon, \ \forall \ \alpha \in Z.$$

W.l.o.g. we can assume $\hat{a} \geq 0$ on $X(A)$ (otherwise replace a with a^2). Then we have

$$(1-\varepsilon)\nu(Z) \leq \int |\hat{a}| d\nu \leq \int \hat{a} d\nu = \int \hat{a} d\mu \leq \int |\hat{a}| d\mu \leq \varepsilon\mu(K),$$

which yields $\nu(Z) \leq \varepsilon\left(\mu(Z) + \nu(Z)\right) < \nu(Z)$ and so a contradiction. Hence, ν is also supported in K and so we have that $\int_K \hat{b} d\mu = \int_K \hat{b} d\nu, \ \forall \ b \in A$. Hence, by Stone-Weierstrass' Theorem 2.3.27, we get $\int_K \varphi d\mu = \int_K \varphi d\nu, \ \forall \ \varphi \in \mathcal{C}(K)$. Then $\mu = \nu$ by the uniqueness in Riesz-Markov-Kakutani Representation Theorem 2.2.5. □