Universität Konstanz
Fachbereich Mathematik und Statistik
Dr. Maria Infusino
Patrick Michalski

# POSITIVE POLYNOMIALS AND MOMENT PROBLEMS–SS 2019

## Exercise Sheet 1

*This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Your solutions will be collected in the lecture on Tuesday 30th of April and discussed in the tutorial on Wednesday the 8th of May (11:45–13:15) in M628.*

**1)** Let $T$ be a preordering in $\mathbb{R}[X]$ and $a \leq b$ in $\mathbb{R}$. Show that if $(X-a)(X-b) \in T$, then $(X-c)(X-d) \in T$ for all $a \leq c \leq d \leq b$.

**Hint:** Reduce to the case $a=-1, b=1$ and show that the polynomial $(X-c)(X-d) - k(X-a)(X-b)$ is psd for some $k \in (0,1)$.

**2)** Let $S_1 := \{X^3, 1-X\}, S_2 := \{X^3 - X\}$ be subsets of $\mathbb{R}[X]$. Show that the preorderings $T_{S_1}$ resp. $T_{S_2}$ are not saturated. Give an alternative description of $K_{S_1}$ resp. $K_{S_1}$ for which the corresponding preordering is saturated.

**3)** Use Lemma 1 to prove the following characterization of saturation of preorderings in $\mathbb{R}[X]$.

Let $S \subseteq \mathbb{R}[X]$ be finite such that $K_S = \bigcup_{i=0}^{n} [a_i, b_i]$ with $b_{i-1} < a_i$ for $i \in \{1, \ldots, n\}$. Then the preordering $T_S$ is saturated if and only if the following two conditions hold:

i) for each $a_i$ there exists $g \in S$ such that $g(a_i) = 0$ and $g'(a_i) > 0$,

ii) for each $b_i$ there exists $g \in S$ such that $g(b_i) = 0$ and $g'(b_i) < 0$.

**Hint:** For the sufficiency show that $T_S$ contains the natural description of $K_S$.

**Lemma 1.** *Let $S \subseteq \mathbb{R}[X]$ be finite such that $K_S$ is compact and $f, g \in \mathrm{Pos}(K_S)$. If $fg \in T_S$ and $f$ and $g$ are relatively prime modulo $T_S \cap -T_S$, then $f \in T_S$ and $g \in T_S$.*

**4)** Let $S_1 := \{X, Y, -Y\}, S_2 := \{X^3, Y, -Y\}, S_3 := \{X^3, 1-X, Y, 1-Y\}$ be subsets of $\mathbb{R}[X, Y]$. Show that the preodering $T_{S_1}$ is saturated, while the preorderings $T_{S_2}$ and $T_{S_3}$ are not saturated.