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POSITIVE POLYNOMIALS AND MOMENT PROBLEMS–SS 2019

Exercise Sheet 2

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 15 near F411 by Tuesday the 14th of May at 10:00. The solutions to this assignment will be discussed in the tutorial on Wednesday the 15th of May (11:45–13:15 in M628).

1) Let $K = \bigcup_{i=0}^{n}[a_i, b_i] \subseteq \mathbb{R}$ with $b_{i-1} < a_i$ for $i \in \{1, \dots, n\}$. Show that:

a) If $a_i < b_i$ for all $i \in \{0, \dots, n\}$, then there exists $f \in \mathbb{R}[X]$ of degree $2(n+1)$ such that $K = K_{\{f\}}$ and $T_{\{f\}}$ is saturated.

b) If $a_i = b_i$ for some $i \in \{0, \dots, n\}$, then there exist $f, g \in \mathbb{R}[X]$ of degree $n+1$ such that $K = K_{\{f,g\}}$ and $T_{\{f,g\}}$ is saturated.

2) Show that:

a) If $f \in \mathbb{R}[X]$ is such that $K_{\{f\}}$ contains an isolated point, then $T_{\{f\}}$ is not saturated.

b) For each $m \in \mathbb{N}$ there exists a bcsas $K \subseteq \mathbb{R}$ such that if $K = K_S$ and T_S is saturated for some finite $S \in \mathbb{R}[X]$, then $|S| \geq m$.

3) Let $S = \{g_1, \dots, g_s\} \subseteq \mathbb{R}[\underline{X}]$ be finite such that M_S is archimedean. Set $B := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, $T' := \{\sum_{i=1}^{n} f_i^2 : n \in \mathbb{N}, f_i \in \mathbb{R}[\underline{X}], f_i > 0 \text{ on } B\}$ and $M' := T' + T'g_1 + \dots + T'g_s$. Show that for $f \in \mathbb{R}[\underline{X}]$ there exists $k \in \mathbb{N}$ such that $k + f \in M'$ using the following steps:

a) If $\sigma \in \sum \mathbb{R}[\underline{X}]^2$ then $l + \sigma \in T'$ for $l \in \mathbb{N}$ sufficiently large.

b) Set $g := 1 + g_1 + \dots + g_s$, then $lg + f \in M'$ for $l \in \mathbb{N}$ sufficiently large.

c) Conclude that $2mg - g^2 \in M'$ for $m \in \mathbb{N}$ sufficiently large and show that this implies $l(m+1)^2 + f \in M'$.

4) Let $S \subseteq \mathbb{R}[X]$ be finite such that $K_S \subseteq \mathbb{R}$ is compact. Show that M_S is archimedean using the following steps:

a) If $n \in \mathbb{N}$ is odd and $f = -X^n + aX^{n-1} + p, g = X^n + aX^{n-1} + q$ for some $a > 0$ and $p, q \in \mathbb{R}[X]_{n-2}$, then $f^2 g + f g^2$ is of even degree and has negative leading coefficient.

b) There exists $f \in M_S$ of even degree with negative leading coefficient.

c) Conclude $k - X^2 \in M_S$ for some $k \in \mathbb{R}$.