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POSITIVE POLYNOMIALS AND MOMENT PROBLEMS–SS 2019

Exercise Sheet 3

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 15 near F411 by Tuesday the 21st of May at 15:00. The solutions to this assignment will be discussed in the tutorial on Wednesday the 29th of May (11:45–13:15 in M628).

1) Let $d \in \mathbb{N}_0$, D a subset of $\mathbb{R}$ consisting of $d+1$ distinct points and set $\Delta := D^n \subseteq \mathbb{R}^n$.

Define $p : \mathbb{R}[\underline{X}]_d \to \mathbb{R}$ by $p(f) := \sum_\alpha |c_\alpha|$ for $f = \sum_\alpha c_\alpha X^\alpha \in \mathbb{R}[\underline{X}]_d$. For $a \in \Delta$ define $q_a : \mathbb{R}[\underline{X}]_d \to \mathbb{R}, f \mapsto q_a(f) := |f(a)|$.

Show that $\mathcal{P} := \{p\}$ and $\mathcal{Q} := \{q_a : a \in \Delta\}$ are families of seminorms which induce the same topology on $\mathbb{R}[\underline{X}]_d$.

2) Show that $\sum \mathbb{R}[\underline{X}]^2$ is closed in $\mathbb{R}[\underline{X}]$ endowed with the finite topology.

Hint: Use Exercise 1) and recall $f \in \sum \mathbb{R}[\underline{X}]^2$ of degree d can be written as a sum of at most $\binom{n+d}{d}$ squares.

3) Let A be a unital commutative $\mathbb{R}$–algebra, $d \in \mathbb{N}$ and $K \subseteq X(A)$ closed.

Define $p(a) := \sup_{\alpha \in K} |\hat{a}(\alpha)|$ for all $a \in A$ and for $\alpha \in K$ define $q_\alpha(a) := |\hat{a}(\alpha)|$ for all $a \in A$. Set $\mathcal{P} := \{p\}$ and $\mathcal{Q} := \{q_\alpha : \alpha \in K\}$.

Show that if K is compact, then $\overline{\sum A^{2d}}^{\tau_\mathcal{P}} = \text{Psd}(K) = \overline{\sum A^{2d}}^{\tau_\mathcal{Q}}$, where $\tau_\mathcal{P}$ (resp. $\tau_\mathcal{Q}$) denotes the lc topology on A induced by $\mathcal{P}$ (resp. $\mathcal{Q}$). Is this result still true if K is non-compact?

4) Let μ be a Radon probability on $\mathbb{R}^n$ and define $L(f) := \int f \mathrm{d}\mu$ for all $f \in \mathbb{R}[\underline{X}]$.

Show that there exist real numbers M and $c_1, \ldots, c_n > 0$ such that $L(X_i^{2d}) \leq c_i^{2d} M$ for all $i \in \{1, \ldots, n\}$ and all $d \in \mathbb{N}$ if and only if $\mu(\prod_{i=1}^n [-c_i, c_i]) = 1$.