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POSITIVE POLYNOMIALS AND MOMENT PROBLEMS–SS 2019

Exercise Sheet 4

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 15 near F411 by Tuesday the 4th of June at 15:00. The solutions to this assignment will be discussed in the tutorial on Wednesday the 12th of June (11:45–13:15 in M628).

1) Let $(s_j)_{j\in\mathbb{N}_0} \subseteq \mathbb{R}$ be the moment sequence of a measure μ on $\mathbb{R}$, i.e. $s_j = \int X^j \,\mathrm{d}\mu$ for all $j \in \mathbb{N}_0$, and denote by δ_x the Dirac measure with mass at $x \in \mathbb{R}$.

Show that if $s_{2j} = 0$ for all $j \in \mathbb{N}$, then $\mu = s_0\delta_0$ and give an example of a sequence $(t_j)_{j\in\mathbb{N}_0} \subseteq \mathbb{R}$ which is not a moment sequence.

2) Show that the real sequences $s := \left(\frac{a^{j+1}}{j+1} + cb^j\right)_{j\in\mathbb{N}_0}$ and $t := \left(\frac{1}{(j+1)(j+2)}\right)_{j\in\mathbb{N}_0}$, where $a, c \geq 0$ and $b \in \mathbb{R}$, are moment sequences by determining a representing measure for each of them.

3) Let $K \subseteq \mathbb{R}$ be closed. Show that the following are equivalent.

(i) If $(s_j)_{j\in\mathbb{N}_0} \subseteq \mathbb{R}$ and $(t_j)_{j\in\mathbb{N}_0} \subseteq \mathbb{R}$ are K–moment sequences, then so is $(s_jt_j)_{j\in\mathbb{N}_0} \subseteq \mathbb{R}$.

(ii) If $x, y \in K$, then $xy \in K$.

4) Let $n \in \mathbb{N}$ and $\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \dots, X_n]$. For $f = \sum_\alpha c_\alpha \underline{X}^\alpha \in \mathbb{R}[\underline{X}]$ define $\|f\|_p := (\sum_\alpha |c_\alpha|^p)^{\frac{1}{p}}$ if $1 \leq p < \infty$ and $\|f\|_\infty := \max_\alpha |c_\alpha|$.

Show that $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_p} = \mathrm{Psd}([-1,1]^n)$ for all $1 \leq p \leq \infty$ without using Theorem 1.3.45.

Hint: Show that if $L \in (\sum \mathbb{R}[\underline{X}]^2)^\vee$, then there exists a $[-1,1]$–representing measure μ for $L\restriction_{\mathbb{R}[X_i]}$ on $\mathbb{R}$ using Exercise 4), Sheet 3. Conclude that $1 - X_i^2 \in \overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_p}$ for all $i \in \{1, \dots, n\}$.