Universität Konstanz
Fachbereich Mathematik und Statistik
Dr. Maria Infusino
Patrick Michalski

POSITIVE POLYNOMIALS AND MOMENT PROBLEMS–SS 2019

Exercise Sheet 5

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 15 near F411 by Wednesday the 19th *of June at 15:00. The solutions to this assignment will be discussed in the tutorial on Wednesday the 26th of June (11:45–13:15 in M628).*

Let $n \in \mathbb{N}$, set $\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$ and let A be a unital commutative $\mathbb{R}$–algebra.

1) Let $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ be linear and $K \subseteq \mathbb{R}^n$ closed. Give necessary and sufficient conditions for the existence of a K–representing Radon measure for L (of course other than $L(\mathrm{Psd}(K)) \subseteq [0, +\infty)$!) for the following choices of n and K

(i) $n = 1$ and $K = K_{S_1}$, where $S_1 := \{4X_1^3 + X_1^2 - 3X_1, 3X_1^4 - 10X_1^3 + 12X_1^2 - 6X_1 + 1\}$,

(ii) $n = 2$ and $K = K_{S_2}$, where $S_2 := \{1 - X_1X_2, X_1 + X_2, -X_1^2 - X_2 + 5\}$.

2) Show that $\left(\frac{1}{n+\pi}\right)_{n\in\mathbb{N}_0}$ is a Stieltjes' moment sequence using localized moment matrices and the following lemma.

Lemma 1. *Let $N \in \mathbb{N}$ and $A := (a_{i,j})_{1\leq i,j\leq N}$ with $a_{i,j} := \frac{1}{x_i - y_j}$, where $x_1, \ldots, x_N \in \mathbb{R}$ (resp. $y_1, \ldots, y_N \in \mathbb{R}$) are pairwise distinct and such that $x_i - y_j \neq 0$ for all $1 \leq i, j \leq N$. Then*

$$\det(A) = \frac{\prod_{i=2}^{N}\prod_{j=1}^{i-1}(x_i - x_j)(y_j - y_i)}{\prod_{i=1}^{N}\prod_{j=1}^{N}(x_i - y_j)}.$$

3) Let $X(A)$ be the character space of A and $\pi : X(A) \to \mathbb{R}^A, \alpha \mapsto (\alpha(a))_{a\in A}$.

Show that $\pi(X(A))$ is closed in $(\mathbb{R}^A, \tau_{\mathrm{prod}})$.

4) Let M be a quadratic module of A and $L \in X(A)$.

Show that there exists a $\mathcal{K}_M$–representing Radon measure for L iff $L(M) \subseteq [0, +\infty)$.