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POSITIVE POLYNOMIALS AND MOMENT PROBLEMS–SS 2019

Exercise Sheet 6

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 15 near F411 by Tuesday the 2nd of July at 15:00. The solutions to this assignment will be discussed in the tutorial on Wednesday the 10th of July (11:45–13:15 in M628).

Let $n \in \mathbb{N}$, set $\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$.

1) Let $I \subseteq \mathbb{R}[\underline{X}]$ be an ideal such that $\mathcal{Z}(I) := \{x \in \mathbb{R}^n : f(x) = 0 \text{ for all } f \in I\}$ is compact and set $A := \mathbb{R}[\underline{X}]/I$. Let T be a finitely generated preordering of A and $L : A \to \mathbb{R}$ linear.

Show that there exists a $\mathcal{K}_T$–representing Radon measure for L iff $L(T) \subseteq [0, +\infty)$.

Hint: Note that if P is a preordering of $\mathbb{R}[\underline{X}]$ and J is an ideal of $\mathbb{R}[\underline{X}]$, then $P + J$ is a preordering of $\mathbb{R}[\underline{X}]$ and $K_{P+J} = K_P \cap \mathcal{Z}(J)$.

2) Set $A := \mathbb{R}[X]/\langle X^2 \rangle$ and endow it with the finest locally convex topology τ. Denote by $X(A)$ the character space of A.

Compute $X(A)$ and show that $\overline{\sum A^2}^{\tau} = \mathrm{Psd}(X(A))$ but $\sum A^2$ is not closed w.r.t. τ.

3) Let $S \subseteq \mathbb{R}[\underline{X}]_1$ be finite such that K_S is non-empty and compact.

Show that there exists a K_S–representing Radon measure for L iff $L(M_S) \subseteq [0, +\infty)$.

4) Let $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ be linear such that $L(\sum \mathbb{R}[\underline{X}]^2) \subseteq [0, +\infty)$. Show that each of the following conditions implies the multivariate Carleman condition.

(i) L is p–continuous for some submultiplicative seminorm p on $\mathbb{R}[\underline{X}]$.

(ii) L is q–continuous, where $q : \mathbb{R}[\underline{X}] \to \mathbb{R}$ is defined by $q(f) := \sum_\alpha (2\lceil |\alpha| / 2 \rceil)! \, |c_\alpha|$ for $f = \sum_\alpha c_\alpha \underline{X}^\alpha \in \mathbb{R}[\underline{X}]$.

(iii) $L(M) \subseteq [0, +\infty)$ for some Archimedean quadratic module M of $\mathbb{R}[\underline{X}]$.