Universität Konstanz
Fachbereich Mathematik und Statistik
Dr. Maria Infusino
Patrick Michalski

POSITIVE POLYNOMIALS AND MOMENT PROBLEMS–SS 2019

Bonus Sheet

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. You may hand in your solutions by Wednesday July 17th at 11:45, when they will be discussed in M628, in order to score bonus points!

1) Let $(s_k)_{k\in\mathbb{N}_0} \subseteq \mathbb{R}$ be such that $s_k > 0$ for all $k \in \mathbb{N}_0$ and $(s_{2k})_{k\in\mathbb{N}_0}$ is log–convex.

Show that

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{s_{2k}}} = \infty \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} \frac{1}{\sqrt[4k]{s_{4k}}} = \infty.$$

2) Prove the following statements.

(i) If $(s_k)_{k\in\mathbb{N}_0}$ is a log–convex sequence of positive real numbers, then so is $(s_{2k})_{k\in\mathbb{N}_0}$.

(ii) If μ is a non-negative Radon measure on $\mathbb{R}$ with finite moments of all orders, then the sequence of even moments of μ is log–convex.

3) Let $(s_k)_{k\in\mathbb{N}_0} \subseteq \mathbb{R}$ be such that $s_k > 0$ for all $k \in \mathbb{N}_0$ and show that the following are equivalent.

(i) $s_k^2 \leq s_{k-1}s_{k+1}$ for all $k \in \mathbb{N}$.

(ii) $\left(\frac{s_k}{s_{k-1}}\right)_{k\in\mathbb{N}}$ is monotone increasing.

(iii) $(\ln(s_k))_{k\in\mathbb{N}}$ is convex.

Recall that a sequence $(t_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ is convex if it is convex as a function $\mathbb{N} \to \mathbb{R}, k \mapsto t_k$.

4) Define

$$f(x) := \tfrac{1}{\sqrt{2\pi}}\chi_{(0,\infty)}(x)x^{-1}\exp\left(-\tfrac{(\ln x)^2}{2}\right) \quad \text{and} \quad g(x) := 1 + \sin(2\pi\ln(x))$$

and let μ be the measure with density f w.r.t. the Lebesgue measure λ on $\mathbb{R}$. Denote by $(s_k)_{k\in\mathbb{N}_0}$ the moment sequence of μ.

Show that

(i) $s_k = \exp\left(\frac{k^2}{2}\right)$ for all $k \in \mathbb{N}_0$,

(ii) $(s_k)_{k\in\mathbb{N}_0}$ does not fulfil the Carleman condition.

(iii) The measure ν with density g w.r.t. μ has the same moment sequence as μ but is not the same measure.