Positive Polynomials and Moment Problems

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Contents

The primary sources for these notes are [37] and [50]. However, I often also took inspiration from [9] and [32].

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Introduction

The main purpose of this course is to explore the fascinating connection existing between positive polynomials and moment problems. The corner stone of this intimate relation is the famous Riesz-Haviland Theorem (proved by Riesz for the one-dimensional case in 1923 and by Haviland for higher dimensions in 1936), which establishes that the problem of characterizing the cone $\mathrm{Psd}(K)$ of all non-negative polynomials on a prescribed subset K of $\mathbb{R}^d$ is the dual facet of the so-called K-moment problem (KMP).

These two problems arose more or less contemporarily at the end of 19th century. In fact, Hilbert's theorem about sum of squares representations of non-negative forms appeared in 1888 and the first formulation of the KMP is due to Stieltjes in 1894, even if moments were already applied by Chebysev, Krein and Markov in the 1880's in studying limit values of integrals. As the characterization of $\mathrm{Psd}(K)$ and the KMP are faces of the same coin, we could start our journey by looking at any of these two problems but, since this course builds up on the contents of the course "Real Algebraic Geometry I" and "Topological Vector Spaces" held during last semester, we are going to start with a quick overview about the main results concerning the fundamental question of characterizing the $\mathrm{Psd}(K)$ cone (e.g. Positivstellensätze, saturation of preorderings and quadratic modules, closure of even power modules, etc.). Then we will rigorously formulate the KMP for $K \subseteq \mathbb{R}^n$ with $n \in \mathbb{N}$, i.e. the problem of establishing whether or not a given sequence of real numbers is the moment sequence of a non-negative Radon measure on K. As Landau brilliantly summarized in [40]: "The moment problem is a classical question in analysis remarkable not only for its own elegance but also for the extraordinary range of subjects theoretical and applied which has illuminated". In this course, we will only discover a small part of the beauty of the moment problem. In particular, after proving the Riesz-Haviland Theorem, we will use it to connect the KMP to the $\mathrm{Psd}(K)$ cone and we will study in detail how the even-power representations/approximations of non-negative polynomials on K influenced the theory of the KMP and at the same time how some of them actually came exactly from the study of the KMP. We will focus only the full finite dimensional KMP for basic closed semi-algebraic sets, i.e. on the case when the starting sequence is infinite and K is a subset of $\mathbb{R}^n$ determined by finitely many polynomial inequalities. Particular attention will be given to the case when K is non-compact which is still open in many of its aspects. Last but not least, we also would like to introduce a very general version of the KMP, namely for linear functionals on any unital commutative real algebra and present some recent results and open problems. Indeed, both the theory of positive polynomials and the moment problem are far to be static and, despite the huge progress of the last 130 years, we can still agree with the statement of Diaconis of 1987 in [40]: "Much is known but still the theory is not up to the demands of the applications" and being motivated to go forward with further research on these topics!

Chapter 1

Positive Polynomials and Sum of Squares

1.1 The ring of multivariate polynomials

Let $n \in \mathbb{N}$. We denote the ring of polynomials in n variables and real coefficients by $\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$. We denote by $0 \in \mathbb{R}[\underline{X}]$ the polynomial with all coefficients equal to zero and by convention we set $\deg(0) = -\infty$. Let us recall some fundamental properties of $\mathbb{R}[\underline{X}]$.

Proposition 1.1.1. *Let $f, g \in \mathbb{R}[\underline{X}]$ s.t. $f \not\equiv 0$ and $g \not\equiv 0$. Then*

(i) $\deg(fg) = \deg(f) + \deg(g)$,

(ii) $\deg(f + g) \leq \max\left\{\deg(f), \deg(g)\right\}$,

(iii) $\deg(f + g) = \max\left\{\deg(f), \deg(g)\right\}$, *if* $\deg(f) \neq \deg(g)$.

Note that $\mathbb{R}[\underline{X}]$ is a real vector space of countable dimension, since a basis is $\{\underline{X}^\alpha \mid \alpha \in \mathbb{N}_0^n\}$ where $\underline{X}^\alpha := X_1^{\alpha_1} \ldots X_n^{\alpha_n}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. In fact, $\mathbb{R}[\underline{X}]$ can be written as a countable union of finite dimensional vector spaces, i.e. $\mathbb{R}[\underline{X}] := \bigcup_{d=0}^{\infty} \mathbb{R}[\underline{X}]_d$ where each $\mathbb{R}[\underline{X}]_d := \{f \in \mathbb{R}[\underline{X}] | \deg(f) \leq d\}$ has dimension $\binom{d+n}{n}$. This structure naturally carries a topology on $\mathbb{R}[\underline{X}]$, which makes it into a Hausdorff topological vector space, namely the *finite topology* τ_f defined by: $U \subseteq \mathbb{R}[\underline{X}]$ is open w.r.t. τ_f iff $\forall d \in \mathbb{N}_0$, $U \cap \mathbb{R}[\underline{X}]_d$ is open in $\mathbb{R}[\underline{X}]_d$ endowed with the euclidean topology (see e.g. [21, Section 4.5] for more details). As we showed in [21, Theorem 4.5.3], τ_f is the finest locally convex topology on $\mathbb{R}[\underline{X}]$ and so every linear functional on $\mathbb{R}[\underline{X}]$ is τ_f-continuous (see [21, Theorem 4.4.3]). This property will be particularly interesting in the study of the n-dimensional moment problem.

Definition 1.1.2. *A polynomial is said to be* homogenous *or* form *if it is the zero polynomial or a linear combination of monomials with same finite degree.*

For $n \in \mathbb{N}$ and $d \in \mathbb{N}_0$, we denote by $\mathcal{F}_{n,d}$ the set of all forms in n variables of degree d, i.e. $\mathcal{F}_{n,d} = \{f \in \mathbb{R}[X_1, \ldots, X_n] \mid f$ is a form and $\deg(f) = d\}$ which is also called set of n-ary d-ics forms.

The set $\mathcal{F}_{n,m}$ is a finite dimensional real vector space of dimension $\binom{d+n-1}{n-1}$.

Definition 1.1.3. *Let $p \in \mathbb{R}[X_1, \ldots, X_n]$ with $\deg(p) = d$. The* homogenization *p_h of p is defined as*

$$p_h(X_0, X_1 \ldots, X_n) := X_0^d \; p\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right).$$

Note that p_h is a homogeneous polynomial of degree d and in $n+1$ variables i.e. $p_h \in \mathcal{F}_{n+1,d}$.

1.2 When is a psd polynomial a sos?

In this section, we are going to consider the fundamental question of when a non-negative polynomial on $\mathbb{R}^n$ can be written as sum of squares of polynomials in $\mathbb{R}[\underline{X}]$.

Definition 1.2.1. *For $p \in \mathbb{R}[\underline{X}]$ we say that*
- *p is* positive semidefinite (psd) *if $p(x) \geq 0$ for all $x \in \mathbb{R}^n$.*
- *p is a* sum of squares (sos) *if $p = \sum_{i=1}^{s} h_i^2$ for some $s \in \mathbb{N}$ and $h_i \in \mathbb{R}[\underline{X}]$ for $i = 1, \ldots, s$.*

We denote by $\mathrm{Psd}(\mathbb{R}^n)$ the cone of all psd polynomials in $\mathbb{R}[\underline{X}]$ and by $\sum \mathbb{R}[\underline{X}]^2$ the cone of all sos of polynomials in $\mathbb{R}[\underline{X}]$.

Clearly, for any $n \in \mathbb{N}$ we always have that $\sum \mathbb{R}[\underline{X}]^2 \subseteq \mathrm{Psd}(\mathbb{R}^n)$. Hence, it is natural to ask for which $n \in \mathbb{N}$ the converse also holds, i.e. when we have that $\mathrm{Psd}(\mathbb{R}^n) = \sum \mathbb{R}[\underline{X}]^2$.

While for $n = 1$, it is easy to show that $\mathrm{Psd}(\mathbb{R}) = \sum \mathbb{R}[X]^2$ (see [37, Proposition 1.2.1]), for $n \geq 2$ it was known already to Hilbert in 1888 that not every psd polynomial is a sos in $\mathbb{R}[\underline{X}]$. Indeed, Hilbert provided a complete characterization of all psd polynomials which are sos. He actually restricted himself only to forms because the property of psd-ness and sos-ness are preserved under homogenization, i.e. for any $p \in \mathbb{R}[\underline{X}]_d$ we have that:
- p is psd iff p_h is psd,
- p is sos iff p_h is sos,

where p_h denotes the homogenization of p (see Definition 1.1.3).

We denote by $\mathcal{P}_{n,d}$ the set of all forms in $\mathcal{F}_{n,d}$ which are psd, and by $\sum_{n,d}$ set of all forms in $\mathcal{F}_{n,d}$ which are sos. It is easy to show that if $p \in \sum_{n,2d}$, then every sos representation of p consists only of homogeneous polynomials of degree d, i.e. $p \in \mathcal{F}_{n,2d},\ p = \sum_{i=1}^{s} p_i^2 \Rightarrow p_i \in \mathcal{F}_{n,d}$.

In [12] Hilbert proved the following result (for a proof see e.g. [31, Lectures 21,22,23], [5, Section 6.3]).

Theorem 1.2.2. $\mathcal{P}_{n,d} = \sum_{n,d}$ *iff*
(i) $n = 2$ *[i.e. binary forms] or*
(ii) $d = 2$ *[i.e. quadratic forms] or*
(iii) $(n, d) = (3, 4)$ *[i.e. ternary quartics].*

The proof of Hilbert was not constructive but in 1927 Motzkin provided the first concrete example of psd form which is not a sos. In addition to Motzkin's form several other examples have been considered. We provide here a short list of the most known ones (for the proofs and references to the original papers see e.g. [31, Lecture 23] and [37, Section 1.2]).

- Motzkin (1927): $z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2 \in \mathcal{P}_{3,6} - \sum_{3,6}$
- Robinson (1969): $x^6 + y^6 + z^6 - (x^4y^2 + x^4z^2 + y^4x^2 + y^4z^2 + z^4x^2 + z^4y^2) + 3x^2y^2z^2 \in \mathcal{P}_{3,6} - \sum_{3,6}$.
- Robinson (1969): $w^4 + x^2y^2 + y^2z^2 + x^2z^2 - 4xyzw \in \mathcal{P}_{4,4} - \sum_{4,4}$
- Choi and Lam (1977): $1 + x^2y^2 + y^2z^2 + z^2x^2 - 4xyz \in \mathcal{P}_{3,6} - \sum_{3,6}$.

Hilbert's theorem and these examples which concretely show that in general $\sum \mathbb{R}[\underline{X}]^2 \subsetneq \mathrm{Psd}(\mathbb{R}^n)$ naturally led to relax the original question and investigate when a psd polynomial can be represented (or approximated) by polynomials whose non-negativity is "more evident", e.g. elements of even power modules of $\mathbb{R}[\underline{X}]$. Actually, the need of looking to these further cones in $\mathbb{R}[\underline{X}]$ becomes even more natural when we analyze the more general question of characterizing the cone of all polynomials in $\mathbb{R}[\underline{X}]$ which are non-negative on a prescribed subset K of $\mathbb{R}^n$.

Definition 1.2.3. *Let $K \subseteq \mathbb{R}^n$. A polynomial $p \in \mathbb{R}[\underline{X}]$ is said to be* positive semidefinite *on K if $p(x) \geq 0$ for all $x \in K$. We denote by* $\mathrm{Psd}(K)$ *the cone of all polynomials which are psd on K, i.e.*

$$\mathrm{Psd}(K) := \{p \in \mathbb{R}[\underline{X}] : p(x) \geq 0, \forall x \in K\}.$$

The following results on polynomials in one variable which are psd on intervals were most probably already known in the early 19th century (see [42] for some discussion on the history of such results) as easy consequences of the fundamental theorem of algebra.

Proposition 1.2.4.
a) $\mathrm{Psd}(\mathbb{R}^+) = \{\sigma_1 + X\sigma_2 : \sigma_1, \sigma_2 \in \sum \mathbb{R}[X]^2\}$.
b) For $a, b \in \mathbb{R}$ with $a < b$, we have
$\mathrm{Psd}([a, b]) = \{\sigma_1 + (b - x)\sigma_2 + (x - a)\sigma_3 : \sigma_1, \sigma_2, \sigma_3 \in \sum \mathbb{R}[X]^2\}$.

Proof. Set $Q := \sum \mathbb{R}[X]^2 + X \sum \mathbb{R}[X]^2$. Clearly, $Q \subseteq \mathrm{Psd}(\mathbb{R}^+)$ and $Q \cdot Q \subseteq Q$. We want to show that $\mathrm{Psd}(\mathbb{R}^+) \subseteq Q$.

Let $0 \not\equiv p \in \mathrm{Psd}(\mathbb{R}^+)$. By the fundamental theorem of algebra, p has the following factorization into irreducibles:

$$p = a \prod_{j=1}^{r} (X - \alpha_j)^{n_j} \prod_{k=1}^{s} \left[(X - u_k)^2 + v_k^2\right]^{\ell_j}, \tag{1.1}$$

where $r, s, n_1, \ldots, n_r, \ell_1, \ldots, \ell_s \in \mathbb{N}$, $a, \alpha_1, \ldots, \alpha_r, u_1, \ldots, u_s, v_1, \ldots, v_s \in \mathbb{R}$ with $\alpha_j \neq \alpha_i$ whenever $j \neq i$ and $\lambda_k := u_k + iv_k$ s.t. $\lambda_k \neq \lambda_i$ and $\lambda_k \neq \overline{\lambda_i}$ whenever $k \neq i$. Since Q is closed under multiplication, it is enough to show that all factors in (1.1) belong to Q. Clearly, $(X - u_k)^2 + v_k^2 \in Q$ and for n_j even also $(X - \alpha_j)^{n_j} \in Q$, so we just need to show that

$$a \prod_{\substack{j \in \{1, \ldots, r\} \\ \text{s.t. } n_j \text{ odd}}}^{r} (X - \alpha_j)^{n_j} \in Q.$$

As $p(x) \geq 0$ for all $x \in \mathbb{R}^+$, we obtain that $a > 0$ by letting $x \to +\infty$ and so $a \in Q$. Also, if α_j is a real root of p with odd multiplicity n_j then p must change sign in a neighbourhood of α_j and so $\alpha_j \leq 0$, which gives in turn that $X - \alpha_j = (-\alpha_j) + X \cdot 1^2 \in Q$. Hence, $p \in Q$.

For a proof of b) see e.g. [50, Proposition 3.3]. □

Proposition 1.2.4 shows that for $K = \mathbb{R}^+$ or $K = [a, b]$ the cone $\mathrm{Psd}(K)$ actually coincides with a certain quadratic module. Let us define such an object for any unital commutative ring.

Definition 1.2.5. *Let A be a commutative ring with 1, $d \in \mathbb{N}$ and denote by $\sum A^{2d}$ the set of all finite sums $\sum a_i^{2d}$, $a_i \in A$.*
a) A $2d$-power module *M in A is a subset $M \subseteq A$ such that $M + M \subseteq M$, $a^{2d} M \subseteq M$ $\forall\, a \in A, 1 \in M$.*
b) A $2d$-power preordering *T in A is a $2d$-power module such that $T \cdot T \subseteq T$.*
In the case $d = 1$, $2d$-power modules (resp., $2d$-power preorderings) are referred to as quadratic modules *(resp.,* quadratic preorderings*)*

Clearly, $\sum A^{2d}$ is a $2d$-power module in A and it is actually the unique smallest one. As $\sum A^{2d}$ is closed under multiplication, we have that $\sum A^{2d}$ is also the unique smallest $2d$-power preordering of A.

Definition 1.2.6. *Let A be a commutative ring with 1 and $d \in \mathbb{N}$. For an arbitrary family $S := \{g_j\}_{j \in J}$ of elements in A (note that J is an arbitrary index set possibly uncountable), the $2d$-power module of A generated by S is defined as*

$$M_S = \left\{\sigma_0 + \sigma_1 g_{j_1} + \ldots + \sigma_s g_{j_s} : s \in \mathbb{N}, j_1, \ldots, j_s \in J, \sigma_0, \ldots, \sigma_s \in \sum A^{2d}\right\}$$

while the $2d$-power preordering of A generated by S as

$$T_S := \left\{ \sum_{e=(e_1,\ldots,e_s)\in\{0,1\}^s} \sigma_e \; g_1^{e_1} \ldots g_s^{e_s} : s \in \mathbb{N}, j_1, \ldots, j_s \in J, \sigma_e \in \sum A^{2d}, \forall e \in \{0,1\}^s \right\}.$$

Note that for a fixed $d \in \mathbb{N}$ and $S \subseteq A$ we have $M_S \subseteq T_S \subseteq \text{Psd}(K_S)$. We say that a module (resp. a preordering) $M \subseteq A$ is *finitely generated* if there exist a finite subset $S \subseteq A$ such that $M = M_S$. For example: ΣA^2 is finitely generated with $S = \emptyset$.

Let us come back now to the ring of polynomials in n variables $\mathbb{R}[\underline{X}]$ and to the question of relating the cone $\text{Psd}(K)$ to even power modules in $\mathbb{R}[\underline{X}]$. In the light of the definitions above, we can restate Proposition 1.2.4 by saying that $\text{Psd}(\mathbb{R}^+)$ coincide with the quadratic preordering generated by $\{X\}$, and that for any $a, b \in \mathbb{R}$ with $a < b$ the cone $\text{Psd}([a, b])$ is the quadratic module generated by $\{b - X, X - a\}$. One can also easily see that $\mathbb{R}^+ = \{x \in \mathbb{R} : p(x) \geq 0\}$ and $[a, b] = \{x \in \mathbb{R} : r(x) \geq 0, q(x) \geq 0\}$ with $p := X, r := b - X, q := X - a$.

This leads us to focus our attention on the special class of closed subsets of $\mathbb{R}^n$ having this same structure.

Definition 1.2.7. *Given $S := \{g_1, \ldots, g_s\} \subset \mathbb{R}[\underline{X}]$, we call the following subset of $\mathbb{R}^n$ the* basic closed semialgebraic set (bcsas) *generated by S:*

$$K_S := \{x \in \mathbb{R}^n : g_i(x) \geq 0, \, i = 1, \ldots, s\}.$$

Hence, we are naturally brought to ask whether for any bcsas K_S of $\mathbb{R}^n$ we have $\text{Psd}(K_S) = T_S$ (resp. $\text{Psd}(K_S) = M_S$), where T_S (resp. M_S) is the quadratic preordering (resp. module) associated to S.

We already know that this is not always true, because for $S = \emptyset$ we have $K_S = \mathbb{R}^n$, $T_S = M_S = \sum \mathbb{R}[\underline{X}]^2$ and we have already seen that for $n \geq 2$ it does not always hold $\text{Psd}(\mathbb{R}^n) = \sum \mathbb{R}[\underline{X}]^2$. However, the results in Proposition 1.2.4 give already a motivation for investigating more deeply the connection between $\text{Psd}(K_S)$, T_S and M_S for finite $S \subset \mathbb{R}[\underline{X}]$.

1.3 Relation between $\text{Psd}(K_S)$ and T_S (resp. M_S)

Fixed a finite subset $S := \{g_1, \ldots, g_s\} \subset \mathbb{R}[X_1, \ldots, X_n]$, we want to study the relation between the (quadratic) preordering associated to S, i.e.

$$T_S := \left\{ \sum_{e=(e_1,\ldots,e_s)\in\{0,1\}^s} \sigma_e \, g_1^{e_1} \ldots g_s^{e_s} : \sigma_e \in \sum \mathbb{R}[\underline{X}]^2, \forall e \in \{0,1\}^s \right\},$$

and $\text{Psd}(K_S)$ where

$$K_S := \{x \in \mathbb{R}^n : g_i(x) \geq 0, \; i = 1, \ldots, s\}.$$

The first result in this direction is the so-called *Stengle Positivstellensatz*, whose proof is due to Stengle in 1974 even if most ideas were already contained in an article of Krivine of 1964.

Theorem 1.3.1. *Let $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[\underline{X}]$ and $f \in \mathbb{R}[\underline{X}]$. Then:*
(1) $f > 0$ on $K_S \Leftrightarrow \exists \; p, q \in T_S$ s.t. $pf = 1 + q$ (Striktpositivstellensatz)
(2) $f \geq 0$ on $K_S \Leftrightarrow \exists \; m \in \mathbb{N}_0, \exists \; p, q \in T_S$ s.t. $pf = f^{2m} + q$ (Nichtnegativstellensatz)
(3) $f = 0$ on $K_S \Leftrightarrow \exists \; m \in \mathbb{N}_0$ s.t. $-f^{2m} \in T_S$ (Real Nullstellensatz)
(4) $K_S = \phi \Leftrightarrow -1 \in T_S$.

Taking $S = \emptyset$ in (2) we obtain an alternative proof for Artin's solution (1927) to the Hilbert's 17th problem posed in 1900 of establishing whether or not a psd polynomial is always a sum of squares of rational functions.

Corollary 1.3.2. *Let $f \in \mathbb{R}[\underline{X}]$.*
If $f(x) \geq 0$ for all $x \in \mathbb{R}^n$ then $f \in \sum \mathbb{R}(\underline{X})^2$.

Proof. Suppose that $f(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $f \not\equiv 0$. By taking $S = \emptyset$ in (2), we get that $\exists \; m \in \mathbb{N}_0, \exists \; p, q \in T_S = \sum \mathbb{R}[\underline{X}]^2$ s.t. $pf = f^{2m} + q$. Since $f \not\equiv 0$, also $f^{2m} + q \not\equiv 0$ and $p \not\equiv 0$. Hence,

$$f = \frac{f^{2m} + q}{p} = \left(\frac{1}{p}\right)^2 p(f^{2m} + q) \in \sum \mathbb{R}(\underline{X})^2.$$

If $f \equiv 0$ then clearly the conclusion holds. □

Theorem 1.3.1-(2) gives a representation of elements in $\text{Psd}(K_S)$ as quotients of elements in T_S. Therefore, it is natural to look for denominator free Positivstellensätze. In particular, in the next subsection we are going to focus on saturation of preorderings, i.e. on the problem of establishing when $\text{Psd}(K_S) = T_S$ holds.

1.3.1 Saturation of preorderings

Definition 1.3.3. *Let $S \subset \mathbb{R}[\underline{X}]$ be finite. The preordering T_S in $\mathbb{R}[\underline{X}]$ is said to be* saturated *if* $\mathrm{Psd}(K_S) = T_S$.

In [31, Lecture 24, 25] the following result was proved in details:

Proposition 1.3.4. *Suppose $n \geq 3$. Let S be a finite subset of $\mathbb{R}[\underline{X}]$ such that $K_S \subseteq \mathbb{R}^n$ and $\mathrm{int}(K_S) \neq \emptyset$. Then there exists $f \in \mathbb{R}[\underline{X}]$ such that $f \geq 0$ on $\mathbb{R}^n$ and $f \notin T_S$.*

This already excludes saturation already for an entire class of preorderings and can be actually obtained as a corollary of the following more general result due to Scheiderer [47, Proposition 6.1].

Theorem 1.3.5. *Let S be a finite subset of $\mathbb{R}[\underline{X}]$ s.t. $\dim(K_S) \geq 3$. Then there exists $f \in \mathbb{R}[\underline{X}]$ s.t. $f \geq 0$ on $\mathbb{R}^n$ and $f \notin T_S$.*

Recall that the dimension of a bcsas $K \subseteq \mathbb{R}^n$ is defined as the Krull dimension of $\frac{\mathbb{R}[\underline{X}]}{\mathcal{I}(K)}$ where $\mathcal{I}(K)$ is the ideal of polynomials vanishing on K. To derive Proposition 1.3.4 from Theorem 1.3.5, it is enough to prove that $\mathrm{int}(K_S) \neq \emptyset$ implies $\dim(K_S) = 3$ (see [31, Lemma 2.7]).

For lower dimensional bcsas, there are examples in which saturation holds and examples in which it fails. An example of one dimensional bcsas which can be described both by a saturated preordering and by a non-saturated preordering is $\mathbb{R}^+$.

Example 1.3.6. *Let $K = [0, +\infty)$. For $S_1 := \{X\}$, we have that $K = K_{S_1}$ and Proposition 1.2.4-a) ensures that $\mathrm{Psd}([0, +\infty)) = T_{S_1}$. Hence, T_{S_1} is saturated. However, by taking the representation $K = K_{S_2}$ with $S_2 := \{X^3\}$, we do not have anymore the saturation of the corresponding preordering. In fact, $X \in \mathrm{Psd}(K)$ but $X \notin T_{S_2}$.*

Suppose that there exist $\sigma_1, \sigma_2 \in \sum \mathbb{R}[X]^2$ s.t. $X = \underbrace{\sigma_1 + X^3\sigma_2}_{=:q}$. Then we have four possibilities:

- *if $\sigma_1 \equiv 0 \equiv \sigma_2$ then $q(X) \equiv 0$.*
- *if $\sigma_1 \equiv 0$ and $\sigma_2 \not\equiv 0$ then $\deg(q)$ is odd and ≥ 3.*
- *if $\sigma_1 \not\equiv 0$ and $\sigma_2 \equiv 0$ then $\deg(q)$ is even.*
- *if $\sigma_1 \not\equiv 0$ and $\sigma_2 \not\equiv 0$ then $\deg(q) = \max\{\deg(\sigma_1), \deg(X^3\sigma_2)\}$ which is either even or odd ≥ 3.*

Hence, $X \not\equiv q$) which leads to the desired contradiction.

In the one variable case, it is possible to show that for any bcsas K of $\mathbb{R}[X]$ there exists $S \subset \mathbb{R}[X]$ finite such that $K = K_S$ and T_S is saturated. Such a S is called the *natural description* of K.

Definition 1.3.7. *Let K be a non-empty bcsas of $\mathbb{R}$, i.e. K is a finite union of intervals and points. The* natural description *of K is defined as the finite subset S_{nat} of $\mathbb{R}[X]$ s.t.*

(i) if $a \in \mathbb{R}$ is the smallest element of K, then $X - a \in S_{nat}$

(ii) if $a \in \mathbb{R}$ is the greatest element of K, then $a - X \in S_{nat}$

(iii) if $a, b \in K$, $a < b$ and $(a, b) \cap K = \phi$, then $(X - a)(X - b) \in S_{nat}$

(iv) no other polynomial is in S_{nat}.

Examples 1.3.8.

- *If $K = [0, +\infty)$ then $S_{nat} = \{X\}$, since 0 is the smallest element of K, K has no greatest element and for all $a, b \in K$ with $a < b$ we have $(a, b) \cap K \neq \emptyset$.*
- *If $K = [0, 1]$ then $S_{nat} = \{X, 1 - X\}$, since 0 is the smallest element of K, 1 is the greatest element of K and for all $a, b \in K$ with $a < b$ we have $(a, b) \cap K \neq \emptyset$.*
- *If $K = -1 \cup [0, 1]$ then $S_{nat} = \{X + 1, 1 - X, X(X + 1)\}$, since -1 is the smallest element of K, 1 is the greatest element of K and $(-1, 0) \cap K = \emptyset$.*

Theorem 1.3.9. *Let K be a non-empty bcsas of $\mathbb{R}$. Then the preordering associated to the natural description S_{nat} of K is saturated.*

Proof. For notational convenience, set S equal to the natural description S_{nat} of K. We want to show that $\mathrm{Psd}(K) = T_S$.

If $K = \mathbb{R}$ then $S = \emptyset$ and $T_S = \sum \mathbb{R}[X]^2$, so the conclusion holds. Therefore, we can assume that $K \subsetneq \mathbb{R}$. Then Definition 1.3.7 provides the following information:

- If K has a smallest element a, then $X - a \in S$ and so

$$\forall d \leq a, X - d = (X - a) \cdot 1^2 + (a - d) \in T_S. \tag{1.2}$$

- if K has a greatest element a, then $a - X \in S$ and so

$$\forall d \geq a, d - X = (a - X) \cdot 1^2 + (d - a) \in T_S. \tag{1.3}$$

- if $a, b \in K$, $a < b$ and $(a, b) \cap K = \phi$, then $(X - a)(X - b) \in S$ and, by Exercise 1 in Sheet 1 we have that

$$\forall d, e \in \mathbb{R} \text{ s.t. } a \leq d \leq e \leq b, (X - d)(X - e) \in T_S. \tag{1.4}$$

Suppose that $f \in \mathrm{Psd}(K)$ and proceed by induction on $\deg(f)$.

If $\deg(f) = 0$ then $f(x) = k$ for all $x \in \mathbb{R}^d$ with $k \geq 0$. Hence, $f \in \sum \mathbb{R}[X]^2 \subset T_S$.

Suppose that $\deg(f) = m \geq 1$ and that for all $g \in \mathrm{Psd}(K)$ with $\deg(g) \leq m-1$ we know that $g \in T_S$. W.l.o.g. we can assume that there exists $c \in \mathbb{R}$ s.t. $f(c) < 0$ (otherwise $f \geq 0$ on $\mathbb{R}$ which gives $f \in \sum \mathbb{R}[X]^2 \subset T_S$). Then there are the following three possibilities: either K has a least element a and $c < a$ or K has a largest element a and $c > a$ or there exist $a, b \in K$ with $a < b$, $(a, b) \cap K = \emptyset$ and $a < c < b$.

Case 1: if K has a least element a and $c < a$, then f has a root d in the interval $(c, a]$. Therefore, $f = (X-d)g$ for some $g \in \mathbb{R}[X]$ with $\deg(g) = m-1$. As $f \geq 0$ on K and $X - d \geq 0$ on K, we get that $g \in \mathrm{Psd}(K)$. Hence, by inductive assumption we have that $g \in T_S$. Also, $X - d \in T_S$ by (1.2) and so $f \in T_S$.

Case 2: If K has a largest element a and $c > a$, then f has a root d in the interval $[a, c)$. Therefore, $f = (d-X)g$ for some $g \in \mathbb{R}[X]$ with $\deg(g) = m-1$. As $f \geq 0$ on K and $d - X \geq 0$ on K, we get that $g \in \mathrm{Psd}(K)$. Hence, by inductive assumption we have that $g \in T_S$. Also, $d - X \in T_S$ by (1.3) and so $f \in T_S$.

Case 3: If there exist $a, b \in K$ with $a < b$, $(a, b) \cap K = \emptyset$ and $a < c < b$, then f has a greatest root d in the interval $[a, c)$ and a least root e in the interval $(c, b]$. Therefore, $f = (X - d)(X - e)g$ for some $g \in \mathbb{R}[X]$ with $\deg(g) = m - 2$. As $f \geq 0$ on K and $(X - d)(X - e) \geq 0$ on K, we get that $g \in \mathrm{Psd}(K)$. Hence, by inductive assumption we have that $g \in T_S$. Also, $(X - d)(X - e) \in T_S$ by (1.4) and so $f \in T_S$. □

Corollary 1.3.10. *Let K be a non-empty bcsas of $\mathbb{R}$. If $S \subset \mathbb{R}[X]$ is finite s.t. $K = K_S$ and $S \supseteq S_{nat}$ (up to a positive scalar multiple factor), then T_S is saturated.*

Proof. By Theorem 1.3.9, we know that $\mathrm{Psd}(K) = T_{S_{nat}}$. As $S \supseteq S_{nat}$ (up to a positive scalar multiple factor), we also have that $T_{S_{nat}} \subseteq T_S$. Hence, $\mathrm{Psd}(K) = T_S$, i.e. T_S is saturated. □

Note that the converse of this result does not hold in general. In fact, if S does not contain the natural description then T_S might be or not be saturated as showed by the following example. However, for non-compact bcsas of $\mathbb{R}$ the converse holds (see Proposition 1.3.12).

Example 1.3.11. *Let $K = [0, 1]$. Then $S_{nat} = \{X, 1 - X\}$ is the natural description of K. Hence, by Theorem 1.3.9, $T_{S_{nat}}$ is saturated. If we take now $S_1 := \{X^3, 1 - X\}$, then $K = K_{S_1}$, S_1 does not contain S_{nat} and T_{S_1} is not saturated (see Sheet 1, Exercise 2 for a proof). However, also $S_2 = \{X(1-X)\}$*

does not contain S_{nat} *and* $K = K_{S_2}$*, but* T_{S_2} *is saturated. Indeed, we have that* $X = X^2 + X(1-X) \in T_{S_2}$ *and* $1 - X = (1-X)^2 + X(1-X) \in T_{S_2}$*, which imply* $T_{S_{nat}} \subseteq T_{S_2}$ *and so that* $\text{Psd}(K) = T_{S_2}$*.*

Proposition 1.3.12. *Let* $K \subseteq \mathbb{R}$ *be a non-compact bcsas of* $\mathbb{R}[X]$ *and* S *a finite subset of* $\mathbb{R}[X]$ *s.t.* $K = K_S$*. Then* T_S *is saturated* $\Leftrightarrow S \supseteq S_{nat}$ *(up to a positive scalar multiple factor).*

Before proving this result, let us introduce the notion of width of a quadratic polynomial in one variable and an elementary related property which will be useful in the proof of Proposition 1.3.12.

Definition 1.3.13. *Let* $f \in \mathbb{R}[X]$ *be such that* $\deg(f) = 2$*. If* r_1, r_2 *are the real roots of* f *and* $r_1 \leq r_2$*, then* width of f *is denoted by* $w(f)$ *and defined to be* $r_2 - r_1$*. If* f *has no real roots, then* $w(f) := 0$*.*

Lemma 1.3.14. *Let* $f_1, f_2 \in \mathbb{R}[X]$ *with* $\deg(f_1) = 2 = \deg(f_2)$ *and positive leading coefficients. Then* $w(f_1 + f_2) \leq \max\{w(f_1), w(f_2)\}$*.*

Proof. W.l.o.g. we can assume that $w(f_1) \geq w(f_2)$ and that $w(f_1) > 0$ (otherwise $w(f_1) = w(f_2) = 0$ and so $f_1 + f_2$ has either one root or no roots, i.e. $w(f_1 + f_2) = 0$). Shifting and scaling we can always reduce to the case $f_1 := X^2 - X$ and $f_2 := c(X - a)(X - (a + b))$ with $a, b, c \in \mathbb{R}$ such that $0 \leq b \leq 1$ and $c > 0$. Thus, we get

$$f_1 + f_2 = (c+1)X^2 - (2ac + bc + 1)X + ca(a + b),$$

whose roots are $\frac{2ac+bc+1 \pm \sqrt{(2ac+bc+1)^2 - 4ca(a+b)(c+1)}}{2(c+1)}$ and so

$$w(f_1 + f_2) = \frac{\sqrt{(2ac + bc + 1)^2 - 4ca(a + b)(c + 1)}}{(c + 1)}.$$

We want to show that $w(f_1 + f_2) \leq w(f_1) = 1$, which by expanding is equivalent to show $(1 - b^2)(c + 1) + (2a + b - 1)^2 \geq 0$. The latter indeed holds since $c > 0$ and $0 \leq b \leq 1$. ∎

Proof. of Proposition 1.3.12.
One direction always holds by Corollary 1.3.10, while for the converse the non-compactness is essential.

Suppose that K_S is not compact and $\text{Psd}(K_S) = T_S$. We can assume that for any $g \in S$ we have $\deg(g) \geq 1$. Since K_S is not compact, it either contains an interval of the form $[c, +\infty)$ or it contains an interval of the form $(-\infty, c]$. Replacing X by $-X$ when necessary in the following proof, we can assume that we are in the first case. This implies that every $g \in S$ is non-negative on $[c, +\infty)$ and so has positive leading coefficient.

Suppose that K_S has a smallest element a and consider $p := X - a$. Then $p \in \mathrm{Psd}(K_S)$ and so by assumption we have $p \in T_S$. This together with the fact that $\deg(p) = 1$ and that $\deg(g) \geq 1$, for all $g \in S$ ensures that $p = \sigma_1 g_1 + \ldots + \sigma_t g_t$, where $\sigma_1, \ldots, \sigma_t \in \mathbb{R}^+$ and $g_i \in S$ with $\deg(g_i) = 1$ for $i = 1, \ldots, t$. As $p(a) = 0$ and $g_i(a) \geq 0$ for all $i = 1, \ldots, t$ (since $a \in K_S$), we can conclude that there exists at least one $i \in \{1, \ldots, t\}$ such that $g_i(a) = 0$. Hence, there exists $r > 0$ such that $g_i = r(X - a)$, i.e. $r(X - a) \in S$ as required.

Suppose now that $a, b \in K_S$ are such that $a < b$ and $(a, b) \cap K_S = \emptyset$ and set $p := (X - a)(X - b)$. Then $p \in \mathrm{Psd}(K_S)$ and so by assumption $p \in T_S$. This together with the fact that $\deg(p) = 2$ and that $\deg(g) \geq 1, \forall g \in S$ ensures that p is a sum of terms of the form σf and ξgh with $\sigma, \xi \in \mathbb{R}^+$ and $f, g, h \in S$ with $\deg(f) \in \{1, 2\}$ and $\deg(g) = 1 = \deg(h)$. Since any linear $g \in S$ is increasing and $g(a) \geq 0$, g is positive on the interval (a, b). Thus, $p \geq \sigma_1 g_1 + \cdots + \sigma_t g_t$ on (a, b), where $\sigma_1, \ldots, \sigma_t \in \mathbb{R}^+ \setminus \{0\}$ and $g_1, \ldots, g_t \in S$ are quadratics which assume at least one negative value on (a, b). Now for each $i \in \{1, \ldots, t\}$, we have that g_i opens upward, $g_i(a) \geq 0$ and $g_i(b) \geq 0$, which imply that g_i has its roots in $[a, b]$ and consequently $w(g_i) \leq b - a$ (see Definition 1.3.13). Since $w(p) = b - a$ and $p \geq \sigma_1 g_1 + \cdots + \sigma_t g_t$ on (a, b), we have that necessarily $w(\sigma_1 g_1 + \cdots + \sigma_t g_t) = b - a$. Hence, by Lemma 1.3.14, we get

$$b - a = w(\sigma_1 g_1 + \cdots + \sigma_t g_t) \leq \max_{i=1,\ldots,t} w(\sigma_i g_i) = \max_{i=1,\ldots,t} w(g_i) \leq b - a,$$

which implies that there exists $i \in \{1, \ldots, t\}$ such that $w(g_i) = b - a$. Hence, g_i necessarily has the form $r(X - a)(X - b)$ for some real $r > 0$, that is, $r(X - a)(X - b) \in S$ as required. □

Applying the so-called Scheiderer's Local Global Principle (see e.g. [37, Section 9]), one can provide examples of two dimensional compact bcsas which can be described by a saturated preordering.

Examples 1.3.15.

1. *The preordering T_S for $S = \{X, 1 - X, Y, 1 - Y\}$ is saturated. Here K_S is the unit square in $\mathbb{R}^2$.*
2. *The preordering T_S for $S = \{1 - X^2 - Y^2\}$ is saturated. Here K_S is the unit disk in $\mathbb{R}^2$.*

However, there are examples of two dimensional compact bcsas for which saturation does not hold.

Example 1.3.16. *Let* $S := \{X^3 - Y^3, 1 - X\}$. *Then* K_S *is compact in* $\mathbb{R}^2$ *and* T_S *is not saturated. Indeed, the polynomial* $X \in \mathbb{R}[X,Y]$ *is nonnegative on* K_S *but does not belong to* T_S.

Suppose that there exists $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \sum \mathbb{R}[X,Y]^2$ *s.t.*

$$X = \underbrace{\sigma_1 + (X^3 - Y^3)\sigma_2 + (1-X)\sigma_3 + (X^3 - Y^3)(1-X)\sigma_4}_{=:q}.$$

Evaluating at $Y = 0$, *we have that* $X \equiv q(X,0) = \sigma_1(X,0) + X^3\sigma_2(X,0) + (1-X)\sigma_3(X,0) + X^3(1-X)\sigma_4(X,0)$, *i.e.* X *belongs to the preordering generated by* $\{X^3, 1-X\}$ *in* $\mathbb{R}[X]$ *which is false as showed in Sheet 1, Exercise 2.*

For non-compact two dimensional bcsas, we have both saturated and non-saturated associated preorderings.

Examples 1.3.17.

1. *If* $S = \emptyset \subset \mathbb{R}[X,Y]$ *then* $T_S = \sum \mathbb{R}[X,Y]^2$ *is not saturated as* $K_S = \mathbb{R}^2$.
2. *If* $S = \{X(1-X)\} \subset \mathbb{R}[X,Y]$, *then* $\mathrm{Psd}(\underbrace{[0,1] \times \mathbb{R}}_{=K_S}) = T_S$,

   i.e. T_S *is saturated (see [38]).*

Summarizing we have that a preordering T_S in $\mathbb{R}[\underline{X}]$ is always not saturated if $\dim(K_S) \geq 3$, but can be or cannot be saturated if $\dim(K_S) \in \{1,2\}$ (depending on the geometry of K_S and the chosen description S).

1.3.2 Representation Theorem and Positivstellensätze

We have seen that saturation of preorderings does not occur for a large class of bcsas. Therefore, in the cases when saturation does not occur, it is still standing our question of how to characterise $\mathrm{Psd}(K_S)$ in terms of T_S without using quotients of its elements. For compact bcsas, a denominator free Positivstellensatz was provided by Schmüdgen in [48] as a corollary of a fundamental result for the $K-$MP for K compact bcsas. This rather surprising result had a great impact in this area and it can be considered a breakthrough in both the theory of positive polynomials and the moment problem. Generalizations of this result were proved by Putinar in [43] and Jacobi in [24] in the coming ten years. Moreover, the Schmüdgen Positivstellensatz gave the impulse to a lively research activity about the moment problem in the non-compact case.

In this section, we are not providing the original Schmüdgen proof but we will derive his Positivstellensatz from a general version of the so-called

Representation Theorem due to Marshall [35]. Actually, Schmüdgen's Positivstellensatz can be obtained as a corollary of a less general and earlier version of the Representation Theorem due to Krivine [29, 30]. This was first noticed by Wörmann in [55], but there was no obvious way to derive Putinar's Positivstellensatz from the Krivine Representation Theorem. Only in 2001 with Jacobi's generalized version of the Representation Theorem [24] it became possible to give a completely algebraic proof of Putinar's Positivstellensatz. The further extension of the Representation Theorem we give here (see Theorem 1.3.24) allows to derive all the above mentioned Positivstellensätze as well as a nice refinement of Putinar's result (see Theorem 1.3.33). In order to state such a Representation Theorem we need to introduce the following general setting.

Let A be a commutative ring with 1 and for simplicity let us assume that $\mathbb{Q} \subseteq A$. We denote by $X(A)$ the *character space* of A, i.e. the set of all unitary ring homomorphisms from A to $\mathbb{R}$. For any $a \in A$, we define the *Gelfand transform* $\hat{a} : X(A) \to \mathbb{R}$ as $\hat{a}(\alpha) := \alpha(a)$, $\forall \alpha \in X(A)$.

For any subset M of A, we set

$$\mathcal{K}_M := \{\alpha \in X(A) : \hat{a}(\alpha) \geq 0,\ \forall a \in M\}.$$

If $M = \sum A^{2d}$ then $\mathcal{K}_M = X(A)$. If M is the $2d-$power module of A generated by $\{p_j\}_{j\in J}$ then $\mathcal{K}_M = \{\alpha \in X(A) : \hat{p_j}(\alpha) \geq 0,\ \forall\ j \in J\}$.

If $a \in M$, then clearly $\hat{a} \geq 0$ on $\mathcal{K}_M$. Does the converse hold, i.e. is it true that if $a \in A$ is such that $\hat{a} \geq 0$ on $\mathcal{K}_M$, then $a \in M$? The Representation Theorem exactly provides an answer to this question. In order to rigorously formulate this result, we need some further notions and properties.

Definition 1.3.18. *A* preprime *of A is a subset T of A such that $T + T \subseteq T$, $T \cdot T \subseteq T$ and $\mathbb{Q}^+ \subseteq T$.*

Definition 1.3.19. *Let T be a preprime of A.*

- *A $T-$*module *of A is a subset M of A such that $M + M \subseteq M$, $T \cdot M \subseteq M$ and $1 \in M$.*
- *A $T-$module is said to be* Archimedean *if for each $a \in A$ there exists $N \in \mathbb{N}$ such that $N \pm a \in M$.*

Remark 1.3.20.
- *A preprime T is itself a $T-$module.*
- *If a preprime T is Archimedian, then any $T-$module is also Archimedian since $T \subseteq M$.*

Note that, for $d \in \mathbb{N}$, any $2d-$power preordering of A is a preprime and any $2d-$power module of A is a $\sum_A^{2d}-$module (see Definition 1.2.5). In particular, any quadratic module of A is a $\sum A^2-$module and the following holds.

Proposition 1.3.21. *Let M be a quadratic module of A. Then*
a) $H_M := \{a \in A : \exists\ N \in \mathbb{N} s.t.\ N \pm a \in M\}$ is a subring of A.
b) $M \cap H_M$ is an Archimedean quadratic module of H_M.
c) M is Archimedean if and only if $H_M = A$.
d) $\forall\ a \in A,\ a^2 \in H_M \Rightarrow a \in H_M$.
e) $\forall\ a_1, \ldots, a_k \in A,\ \sum_{i=1}^{k} a_i^2 \in H_M \Rightarrow a_i \in H_M\ \forall\ i = 1, \ldots, k$.

Proof. (see e.g. [31, Proposition 2.1, Lecture 28] and [37, Proposition 5.2.3]) □

Corollary 1.3.22. *Let M be a quadratic module of $\mathbb{R}[\underline{X}]$. The following are equivalent:*
(1) M is Archimedean
(2) $\exists\ N \in \mathbb{N}$ such that $N - \sum_{i=1}^{n} X_i^2 \in M$.
(3) $\exists\ N \in \mathbb{N}$ such that $N \pm X_i \in M$ for $i = 1, \ldots, n$.

Proof.
(1)⇒(2) This is clear from the definition of Archimedean quadratic module.
(2)⇒(3) Suppose that there exists $k \in \mathbb{N}$ such that $k - \sum_{i=1}^{n} X_i^2 \in M$. Then $\sum_{i=1}^{n} X_i^2 \in H_M$ and so Proposition 1.3.21-e) ensures that for each $i \in \{1, \ldots, n\}$ we have that $X_i \in H_M$. Hence, there exists $N \in \mathbb{N}$ such that $N \pm X_i \in M$, i.e. (3) holds.
(3)⇒(1) Suppose that there exists $k \in \mathbb{N}$ such that $k \pm X_i \in M$ for $i = 1, \ldots, n$. Then $X_i \in H_M$ for $i = 1, \ldots, n$ and since $\mathbb{R}^+ \subseteq M$ we also have that $\mathbb{R} \subseteq H_M$. Hence, Proposition 1.3.21-a) guarantees that $H_M = \mathbb{R}[\underline{X}]$, which is equivalent to the Archimedeanity of M by Proposition 1.3.21-c). □

For the general version of the Representation Theorem, we need to strengthen a bit our assumptions on T.

Definition 1.3.23. *A preprime T is said to be* weakly torsion *if for any $a \in A$ there exists a positive rational r and $m \in \mathbb{N}$ such that $(r+a)^m \in T$.*

Clearly, any Archimedean preprime is weakly torsion. Also, for $d \in \mathbb{N}$, any $2d-$power preordering of A is a weakly torsion preprime (just take $m = 2d$).

We are finally ready to state the general version of the Representation Theorem we had announced (for a proof see [35], [24] and [37, Theorem 5.4.4]). Other versions of the Representation Theorem can be found in [1], [27], [29, 30], [11], [54].

Theorem 1.3.24. *Let A be a commutative ring with 1 such that $\mathbb{Q} \subseteq A$. If T is a weakly torsion preprime of A and M an Archimedean $T-$module of A, then for any $a \in A$ we have:*

$$\hat{a} > 0 \text{ on } \mathcal{K}_M \Rightarrow a \in M.$$

Remark 1.3.25.

a) Taking $M = T$ with T Archimedean preprime, we get Krivine's version of the Representation Theorem (see [29, 30] and also [31, Corollary 2.1, Lecture 27]). From this version, we can already derive the Schmüdgen Positivstellensatz as it was first noted by Wörmann in [55] (see Theorem 1.3.31).

b) Taking $d \in \mathbb{N}$ and $T = \sum A^{2d}$, we get the Jacobi-Prestel Positivstellensatz (see Theorem 1.3.28) from which one can straightforwardly derive Putinar's Positivstellensatz (see Theorem 1.3.29).

To understand the meaning of the Representation Theorem 1.3.24 for $A = \mathbb{R}[X_1, \dots, X_n]$, we need to understand what the Gelfand transform and the characters are in this special case.

Proposition 1.3.26.

a) The identity $id : \mathbb{R} \to \mathbb{R}$ is the unique ring homomorphism from $\mathbb{R}$ to $\mathbb{R}$

b) $X(\mathbb{R}[X_1, \dots, X_n])$ is in a one-to-one correspondence with $\mathbb{R}^n$.

Proof.

a) Suppose that $\alpha : \mathbb{R} \to \mathbb{R}$ is a ring homomorphism such that $\alpha \neq id$. Then there exists $a \in \mathbb{R}$ such that $\alpha(a) \neq id(a)$, say $\alpha(a) < id(a)$. Thus, there exists $q \in \mathbb{Q}$ such that $\alpha(a) < q < id(a)$ and so $\alpha(a-q) < 0$ while $id(a-q) > 0$, i.e. $a - q \notin \alpha^{-1}(\mathbb{R}^+)$ and $a - q \in id^{-1}(\mathbb{R}^+)$. Hence, we have that $\alpha^{-1}(\mathbb{R}^+) \neq id^{-1}(\mathbb{R}^+)$. However, α maps squares to squares and so we also have that $\alpha^{-1}(\mathbb{R}^+) = \mathbb{R}^+ = id^{-1}(\mathbb{R}^+)$, which yields a contradiction.

b) By a), for any $\alpha \in X(\mathbb{R}[X_1, \dots, X_n])$ we have that $\alpha \restriction_{\mathbb{R}} = id$, which easily implies that α is completely determined by $(\alpha(X_1), \dots, \alpha(X_n)) \in \mathbb{R}^n$.

In fact, for any $p := \sum_\beta p_\beta \underline{X}^\beta \in \mathbb{R}[\underline{X}]$ with $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ and $\underline{X}^\beta := X_1^{\beta_1} \cdots X_n^{\beta_n}$, we have that

$$\begin{aligned}\alpha(p) &= \alpha\left(\sum_\beta p_\beta \underline{X}^\beta\right) = \sum_\beta \alpha(p_\beta)\alpha(X_1)^{\beta_1}\cdots\alpha(X_n)^{\beta_n}\\ &= \sum_\beta p_\beta \alpha(X_1)^{\beta_1}\cdots\alpha(X_n)^{\beta_n} = p\left(\alpha(X_1),\ldots,\alpha(X_n)\right).\end{aligned}$$

Conversely, for any $y \in \mathbb{R}^n$ we can define the map $\alpha_y : \mathbb{R}[X_1, \ldots, X_n] \to \mathbb{R}$ by $\alpha_y(p) := p(y)$ for any $p \in \mathbb{R}[\underline{X}]$, which is clearly a ring homomorphism, i.e. $\alpha_y \in X(\mathbb{R}[X_1, \ldots, X_n])$. Hence, $X(\mathbb{R}[X_1, \ldots, X_n]) \cong \mathbb{R}^n$. □

Remark 1.3.27. *Using the isomorphism between $X(\mathbb{R}[X_1, \ldots, X_n])$ and $\mathbb{R}^n$ we get that for any $p \in \mathbb{R}[\underline{X}]$ the Gelfand transform $\hat{p}$ is identified with the polynomial p itself. Moreover, if M is a $2d-$power module of $\mathbb{R}[\underline{X}]$ then*

$$\begin{aligned}\mathcal{K}_M &= \{\alpha \in X(\mathbb{R}[X_1,\ldots,X_n]) : \hat{q}(\alpha) \geq 0,\ \forall q \in M\}\\ &\cong \{x \in \mathbb{R}^n : q(x) \geq 0, \forall q \in M\} = K_M.\end{aligned}$$

In particular, if $S \subset \mathbb{R}[X_1, \ldots, X_n]$ and M_S is the $2d-$power module generated by S then

$$\begin{aligned}\mathcal{K}_{M_S} \cong K_{M_S} &= \{x \in \mathbb{R}^n : q(x) \geq 0, \forall q \in M_S\}\\ &= \{x \in \mathbb{R}^n : q(x) \geq 0, \forall q \in S\} = K_S.\end{aligned}$$

Applying the Representation theorem 1.3.24 for $T = \sum \mathbb{R}[\underline{X}]^{2d}$ with $d \in \mathbb{N}$ and using Remark 1.3.27 we easily get the following results.

Theorem 1.3.28 (Jacobi-Prestel's Positivstellensatz)**.** *Let M be an Archimedean $2d-$power module of $\mathbb{R}[\underline{X}]$ with $d \in \mathbb{N}$. Then for any $f \in \mathbb{R}[\underline{X}]$*

$$f > 0 \text{ on } K_M \Rightarrow f \in M.$$

In particular, taking $d = 1$ we easily get:

Theorem 1.3.29 (Putinar's Positivstellensatz)**.** *Let $S \subset \mathbb{R}[\underline{X}]$ be finite such that the quadratic module M_S generated by S is Archimedean. Then for any $f \in \mathbb{R}[\underline{X}]$*

$$f > 0 \text{ on } K_S \Rightarrow f \in M_S.$$

To get Schmüdgen's Positivstellensatz from Theorem 1.3.24, we need to understand how the compactness of K_S relates to the Archimedeanity of the associated quadratic preordering T_S. The following criterion was provided by Wörmann in [55].

Theorem 1.3.30 (Wörmann Theorem)**.** *Let $S \subset \mathbb{R}[\underline{X}]$ be finite. The corresponding bcsas K_S is compact if and only if the associated quadratic preordering T_S is Archimedean.*

Proof. (see e.g. [37, Theorem 6.1.1] or [31, Theorem 2.1, Lecture 28]) □

Theorem 1.3.31 (Schmüdgen's Positivstellensatz)**.** *Let $S \subset \mathbb{R}[\underline{X}]$ be finite such that the associated bcsas K_S is compact. Then for any $f \in \mathbb{R}[\underline{X}]$*

$$f > 0 \text{ on } K_S \Rightarrow f \in T_S.$$

Proof. By Wörmann Theorem, the quadratic preordering T_S is Archimedean and so a weakly torsion preprime. Hence, by taking $T = M = T_S$ in the Representation Theorem 1.3.24 and using Remark 1.3.27, we obtain the conclusion. □

Remark 1.3.32.

a) *Schmüdgen's Positivstellensatz fails in general if we drop the compactness assumption on K_S.*

   For example,
   - *for $n = 1$ and $S = \{X^3\}$, we have that $K_S = [0, \infty)$ is non-compact and $X + 1 > 0$ on K_S but $X + 1 \notin T_S$ (otherwise there would exist $\sigma_0, \sigma_1 \in \sum \mathbb{R}[X]^2$ such that $X + 1 = \sigma_0 + \sigma_1 X^3$ but this impossible as the right-hand side would have either even degree or odd degree ≥ 3 (see Example 1.3.6)).*
   - *for $n = 2$ and $S = \emptyset$, we have that the strictly positive version of the Motzkin polynomial $1 - X_1^2X_2^2 + X_1^2X_2^4 + X_1^4X_2^2$ is indeed strictly positive on $K_S = \mathbb{R}^2$ but does not belong to $T_S = \sum \mathbb{R}[X_1, X_2]^2$.*

b) *Schmüdgen's Positivstellensatz fails in general if the assumption of strict positivity on K_S is replaced by the nonnegativity on K_S. For example, for $n = 1$ and $S = \{(1 - X^2)^3\}$ we have that $K_S = [-1, 1]$ is compact, $1 - X^2 \geq 0$ on K_S but $1 - X^2 \notin T_S$.*

c) *Schmüdgen's Positivstellensatz fails in general when the preordering T_S is replaced by the quadratic module M_S. The question of whether this was true was first posed by Putinar in [43] and got a negative answer in [25,* *Example 4.6], where Jacobi and Prestel showed that for* $n \geq 2$ *and* $S = \{g_1, \ldots, g_{n+1}\}$ *with* $g_i := X_i - \frac{1}{2}$ *for* $i = 1, \ldots, n$ *and* $g_{n+1} := 1 - \prod_{i=1}^n X_i$ *we have that* K_S *is compact but* M_S *is not Archimedean (thus, there exists* $N \in \mathbb{N}$ *such that* $N - \sum_{i=1}^n X_i^2 > 0$ *on* K_S *but* $N - \sum_{i=1}^n X_i^2 \notin M_S$*). This counterexample provides a general negative answer to Putinar's question, but there are actually cases in which the compactness of* K_S *implies the Archimedeanity of* M_S*. For instance, this holds in each of the following cases*

- $|S| = 1$ *(as in this case* $T_S = M_S$*)*
- $|S| = 2$ *(proof in [25]).*
- $n = 1$ *(proof in Sheet 2, Exercise 2)*
- S *consists only of linear polynomials (see [37, Theorem 7.1.3]).*

Note that if M_S *is Archimedean then* K_S *is always compact. Indeed, Archimedeanity of* M_S *implies that there exists* $N \in \mathbb{N}$ *such that* $N - \sum_{i=1}^n X_i^2 \in M_S$ *and so* $N - \sum_{i=1}^n X_i^2 \geq 0$ *on* K_S*. Hence,* K_S *is contained in the closed ball of radius* $\sqrt{N}$ *in* $\mathbb{R}^n$ *endowed with the euclidean topology, i.e.* K_S *is bounded. This together with the fact that* K_S *is closed provides that* K_S *is compact.*

Let us give now a further application of the Representaton Theorem 1.3.24, which shows the power of this very general version and allows to refine the representation provided by Putinar's Positivstellensatz (see Theorem 1.3.29).

Theorem 1.3.33. *Let* $S := \{g_1, \ldots, g_s\}$ *be a finite subset of* $\mathbb{R}[\underline{X}]$ *such that the associated quadratic module* M_S *is Archimedean. Then, for any real* $N > 0$*, any* $f > 0$ *on* K_S *can be represented as* $f = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s$ *where each* σ_i *is a sum of squares of polynomials which are strictly positive on the closed ball* $B_N := \{x \in \mathbb{R}^n : \|x\| \leq N\}$ *(here* $\|\cdot\|$ *is the euclidean norm).*

Proof. Let N be a strictly positive real number and $f > 0$ on K_S. Define

$$\tilde{T}^* := \{\sum f_i^2 : f_i \in \mathbb{R}[\underline{X}], f_i > 0 \text{ on } B_N\}, \tilde{T} := \tilde{T}^* \cup \{0\}$$

and

$$\tilde{M}^* := \tilde{T}^* + \tilde{T}^* g_1 + \cdots + \tilde{T}^* g_s, \tilde{M} := \tilde{M}^* \cup \{0\}.$$

As B_N is compact, for any $g \in \mathbb{R}[\underline{X}]$ there exists $r \in \mathbb{Q}$ positive such that $r + g > 0$ on B_N and so $(r+g)^2 \in \tilde{T}^*$. Hence, $\tilde{T}$ is a weakly torsion preprime.
Claim: For any $h \in \mathbb{R}[\underline{X}]$ there exists $l \in \mathbb{N}$ such that $l + h \in \tilde{M}^*$.
(see Sheet 2, Exercise 3 for a proof of the Claim).

Since $\tilde{T}$ is a preprime, it easily follows from the definitions that $\tilde{M} + \tilde{M} \subseteq \tilde{M}$ and $\tilde{T}\tilde{M} \subseteq \tilde{M}$. Moreover, applying the claim for $h = 0$, we have that there exists $l \in \mathbb{N}$ such that $l \in \tilde{M}^*$ and so $1 = l \cdot \frac{1}{l} \in \mathbb{Q}\tilde{M}^* \subseteq \tilde{M}^* \subseteq \tilde{M}$. Thus, $\tilde{M}$ is a $\tilde{T}-$module. By the Claim, $\tilde{M}$ is also Archimedean.

To apply Theorem 1.3.24, it remains to show that $K_S = K_{\tilde{M}}$. Once this is proved, the theorem ensures that $f \in \tilde{M}$.

($\subseteq$) As $\tilde{M} \subseteq M_S$, we have that $K_S \subseteq \{x \in \mathbb{R}^n : g(x) \geq 0, \forall g \in \tilde{M}\} = K_{\tilde{M}}$.

($\supseteq$) Suppose there exists $x \in K_{\tilde{M}}$ such that $x \notin K_S$. Then there exists $i \in \{1, \ldots, s\}$ such that $g_i(x) < 0$. Take $h := \sum_{j=0}^{s} r_j g_j$ with $g_0 := 1$, $r_j = 1$ for all $j \neq i$, and $r_i > ls$ where $l \in \mathbb{N}$ such that $g_j(x) < -l g_i(x)$ for all $j \neq i$. Thus, $h \in \tilde{M}$ but $h(x) = \sum_{j \neq i} g_j(x) + r_i g_i(x) < -lsg_i(x) + r_i g_i(x) = (r_i - ls) g_i(x) < 0$, which yields $x \notin K_{\tilde{M}}$ that is a contradiction. □

1.3.3 Closure of even power modules

In this section, we are going to see how the Positivstellensätze considered in the previous section can be used to better understand the relation between Psd(K_S) and T_S (resp. M_S). For this purpose, let us recall the following application of Hahn-Banach Theorem which we have studied in [21, Section 5.2].

Corollary 1.3.34. *Let (X, τ) be a locally convex t.v.s. over the real numbers. If C is a nonempty closed cone of X and x and $x_0 \in X \setminus C$, then there exists a linear $\tau-$continuous functional $L : X \to \mathbb{R}$ non identically zero s.t. $L(C) \geq 0$ and $L(x_0) < 0$.*

Recall that a cone of X is a subset $C \subseteq X$ such that $C + C \subseteq C$ and $\lambda C \subseteq C$ for all $\lambda \in \mathbb{R}^+$.

Proof. As C is closed in (X, τ) and $x_0 \in X \setminus C$, we have that $X \setminus C$ is an open neighbourhood of x_0. Then the local convexity of (X, τ) guarantees that there exists an open convex neighbourhood V of x_0 s.t. $V \subseteq X \setminus C$ i.e. $V \cap C = \emptyset$. By the Geometric form of Hahn-Banach theorem, we have that there exists a closed hyperplane H of X separating V and C, i.e. there exists $L : X \to \mathbb{R}$ linear $\tau-$continuous and not identically zero s.t. $L(C) \geq 0$ and $L(V) < 0$ (see [21, Proposition 5.2.1-c)] for more details). In particular, $L(C) \geq 0$ and $L(x_0) < 0$. □

Given a convex cone C in any t.v.s. (X, τ) we define the first and the second dual of C w.r.t. τ respectively as follows:

$$C_\tau^\vee := \{\ell : X \to \mathbb{R} \text{ linear} \,|\, \ell \text{ is } \tau-\text{continuous and } \ell(C) \geq 0\}$$

$$C_\tau^{\vee\vee} := \{x \in X \,|\, \forall\, \ell \in C_\tau^\vee,\, \ell(x) \geq 0\}.$$

Note that

- $C \subseteq C_\tau^{\vee\vee}$, because if $x \in C$ then for all $\ell \in C_\tau^\vee$ we have $\ell(x) \geq 0$ by definition of $C_\tau^\vee$.
- $C_\tau^{\vee\vee}$ is closed in (X, τ), because $C_\tau^{\vee\vee} = \bigcap_{\ell \in C_\tau^\vee} \ell^{-1}([0, \infty))$ and each $\ell \in C_\tau^\vee$ is by definition $\tau-$continuous.

Hence, $\overline{C}^\tau \subseteq C_\tau^{\vee\vee}$ always holds.

Corollary 1.3.35. *Let (X, τ) be a locally convex t.v.s. over the real numbers. If C is a nonempty convex cone in X, then $\overline{C}^\tau = C_\tau^{\vee\vee}$.*

Proof. Suppose there exists $x_0 \in C_\tau^{\vee\vee} \setminus \overline{C}^\tau$. By Corollary 1.3.34, there exists a $\tau-$continuous functional $L : X \to \mathbb{R}$ non identically zero s.t. $L(\overline{C}^\tau) \geq 0$ and $L(x_0) < 0$. As $L(C) \geq 0$ and L is $\tau-$continuous, we have $L \in C_\tau^\vee$. This together with the fact that $L(x_0) < 0$ give $x_0 \notin C_\tau^{\vee\vee}$, which is a contradiction. Hence, $\overline{C}^\tau = C_\tau^{\vee\vee}$. □

The previous results clearly apply to $\mathbb{R}[\underline{X}]$ endowed with the finite topology τ_f. Indeed, we have already observed in Section 1.1 that τ_f is actually the finest locally convex topology on $\mathbb{R}[\underline{X}]$ and so that $(\mathbb{R}[\underline{X}], \tau f)$ is a locally convex t.v.s.. Moreover, keeping in mind [21, Theorem 3.1.1], it is easy to prove that $(\mathbb{R}[\underline{X}], \tau_f)$ is actually a topological algebra, i.e. a t.v.s. with separately continuous multiplication. Hence, we can prove the following properties.

Proposition 1.3.36. *Let $d \in \mathbb{N}$, M a $2d-$power module of $\mathbb{R}[\underline{X}]$ and φ the finest locally convex topology on $\mathbb{R}[\underline{X}]$. Then*
(a) $\overline{M}^\varphi$ is a $2d-$power module of $\mathbb{R}[\underline{X}]$
(b) If M is a preordering, then $\overline{M}^\varphi$ is a preordering.
(c) $\overline{M}^\varphi = M_\varphi^{\vee\vee} \subseteq \mathrm{Psd}(K_M)$

Proof. (a) As M is a $2d-$power module of $\mathbb{R}[\underline{X}]$ and $(\mathbb{R}[\underline{X}], \varphi)$ is a topological algebra, we have that $1 \in M \subseteq \overline{M}^\varphi$, $\overline{M}^\varphi + \overline{M}^\varphi \subseteq \overline{M + M}^\varphi \subseteq \overline{M}^\varphi$ and $p^{2d}\overline{M}^\varphi \subseteq \overline{p^{2d}M}^\varphi \subseteq \overline{M}^\varphi$. Hence, $\overline{M}^\varphi$ is a $2d-$power module.

(b) If M is a $2d-$power preordering, then (a) ensures that $\overline{M}^\varphi$ is a $2d-$power module. Moreover, using that $M \cdot M \subseteq M$ and $(\mathbb{R}[\underline{X}], \varphi)$ is a topological algebra, we get that $\overline{M}^\varphi \cdot \overline{M}^\varphi \subseteq \overline{M \cdot M}^\varphi \subseteq \overline{M}^\varphi$. Hence, $\overline{M}^\varphi$ is a preordering.

(c) Since every $2d-$power module is a cone, Corollary 1.3.35 guarantees that $\overline{M}^\varphi = M_\varphi^{\vee\vee}$. For any $x \in \mathbb{R}^n$, the map $e_x : \mathbb{R}[\underline{X}] \to \mathbb{R}$ defined by $e_x(p) := p(x)$ is a $\mathbb{R}-$algebra homomorphism. Hence, for all $x \in \mathbb{R}^n$, e_x is linear and so $\varphi-$continuous. Also, for all $x \in K_M$, we have that $e_x(g) = g(x) \geq 0$ for all $g \in M$, i.e. $e_x \in M_\varphi^\vee$. Then for any $f \in M_\varphi^{\vee\vee}$ we get that $f(x) = e_x(f) \geq 0$ for all $x \in K_M$, i.e. $f \in \mathrm{Psd}(K_S)$. □

Let us now come back to the Positivstellensätze introduced in the last sections and derive from them the corresponding Nichtnegativstellensätze.

Corollary 1.3.37 (Jacobi-Prestel's Nichtnegativstellensatz)**.** *Let M be an Archimedean $2d-$power module of $\mathbb{R}[\underline{X}]$ with $d \in \mathbb{N}$. Then for any $f \in \mathbb{R}[\underline{X}]$*

$$f \geq 0 \text{ on } K_M \Rightarrow \forall \varepsilon > 0, f + \varepsilon \in M.$$

Proof. Let $f \in \mathbb{R}[\underline{X}]$ be such that $f \geq 0$ on K_M. Then for any $\varepsilon > 0$, we have that $f + \varepsilon > 0$ on K_M and so Theorem 1.3.28 ensures that $f + \varepsilon \in M$. □

Corollary 1.3.38 (Putinar's Nichtnegativstellensatz)**.** *Let $S \subset \mathbb{R}[\underline{X}]$ be finite such that the quadratic module M_S generated by S is Archimedean. Then for any $f \in \mathbb{R}[\underline{X}]$*

$$f \geq 0 \text{ on } K_S \Rightarrow \forall \varepsilon > 0, f + \varepsilon \in M_S.$$

Corollary 1.3.39 (Schmüdgen's Nichtnegativstellensatz)**.** *Let $S \subset \mathbb{R}[\underline{X}]$ be finite such that the associated bcsas K_S is compact. Then for any $f \in \mathbb{R}[\underline{X}]$*

$$f \geq 0 \text{ on } K_S \Rightarrow \forall \varepsilon > 0, f + \varepsilon \in T_S.$$

Using Proposition 1.3.36 and the Nichtnegativstellensätze, we easily obtain the following closure results.

Corollary 1.3.40. *Let M be an Archimedean $2d-$power module of $\mathbb{R}[\underline{X}]$ with $d \in \mathbb{N}$. Then $\mathrm{Psd}(K_M) = \overline{M}^{\varphi}$.*

Proof. By Proposition 1.3.36-(c), $\mathrm{Psd}(K_M) \supseteq \overline{M}^{\varphi}$. For proving the converse inclusion, let us consider $f \in \mathrm{Psd}(K_M)$ and $\varepsilon > 0$. The Jacobi-Prestel's Nichtnegativstellensatz 1.3.37 guarantees that $f + \varepsilon \in M$ and so, for any $\ell \in M_{\varphi}^{\vee}$, we have that $\ell(f + \varepsilon) \geq 0$, i.e. $\ell(f) \geq -\varepsilon\ell(1)$. Then $\ell(f) \geq 0$ and so $f \in M_{\varphi}^{\vee\vee} \overset{\text{Cor1.3.35}}{=} \overline{M}^{\varphi}$. □

Corollary 1.3.41. *Let $S \subset \mathbb{R}[\underline{X}]$ be finite such that the quadratic module M_S generated by S is Archimedean. Then $\mathrm{Psd}(K_S) = \overline{(M_S)}^{\varphi}$*

Corollary 1.3.42. *Let $S \subset \mathbb{R}[\underline{X}]$ be finite such that the associated bcsas K_S is compact. Then $\mathrm{Psd}(K_S) = \overline{(T_S)}^{\varphi}$*

These results make us understanding that even when we do not have saturation of the preordering we still have cases when $\mathrm{Psd}(K_S)$ can be characterized in terms of T_S or M_S, namely as closures of these cones w.r.t. the finest locally convex topology φ. Note that typically T_S is not closed.

In fact, if S is a finite subset of $\mathbb{R}[\underline{X}]$ such that K_S is compact and $\dim(K_S) \geq 3$, then Corollary 1.3.42 ensures that $\mathrm{Psd}(K_S) = \overline{(T_S)}^{\varphi}$ but by Theorem 1.3.5 we also know that $\mathrm{Psd}(K_S) \neq T_S$ so $T_S \neq \overline{(T_S)}^{\varphi}$, i.e.$T_S$ is not closed in $(\mathbb{R}[\underline{X}], \varphi)$. In the case when K_S is not compact (and so M_S is not Archimedean), we cannot apply the previous closure results, so is it natural to ask if we can get similar results by considering closures w.r.t. other topologies rather than φ.

Closures of even power modules of $\mathbb{R}[X_1, \ldots, X_n]$ have been studied already since the seventies. Indeed, the cone $\sum \mathbb{R}[X_1, \ldots, X_n]^2$ is closed in $(\mathbb{R}[\underline{X}], \varphi)$ (see Sheet 3, Exercise 2), so taking its closure w.r.t. φ does not help to characterize $\mathrm{Psd}(\mathbb{R}^n)$ for $n \geq 2$ (as $\mathrm{Psd}(\mathbb{R}^n) \neq \sum \mathbb{R}[\underline{X}]^2 = \overline{\sum \mathbb{R}[\underline{X}]^2}^{\varphi}$). However, every polynomial in $\mathrm{Psd}(\mathbb{R}^n)$ can be approximated by elements in $\sum \mathbb{R}[\underline{X}]^2$ w.r.t. the topology induced by the norm $\|\cdot\|_1$, where $\|f\|_1 := \sum_\alpha |f_\alpha|$ for any $f = \sum_\alpha f_\alpha \underline{X}^\alpha \in \mathbb{R}[\underline{X}]$. In fact, in [2, Theorem 9.1] the authors show that $\mathrm{Psd}([-1,1]^n) = \overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_1}$, i.e. $\sum \mathbb{R}[\underline{X}]^2$ is dense in $\mathrm{Psd}([-1,1]^n)$ w.r.t. $\|\cdot\|_1$ on $\mathbb{R}[\underline{X}]$ (see also [34]). This result is actually established in [2] as a corollary of a general result valid for any commutative semigroup. In [3] and [4] the results in [2] were extended further, to include commutative semigroups with involution and topologies induced by absolute values. In [15] a new proof of these results is given by using the Representation Theorem 1.3.24 and they are at the same time extended from sums of squares to sums of $2d-$powers. In particular, the authors prove that for any $d \in \mathbb{N}$ we get $\mathrm{Psd}([-1,1]^n) = \overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\|\cdot\|_1}$. The closure of $\sum \mathbb{R}[\underline{X}]^{2d}$ w.r.t. to $\|\cdot\|_p$ with $1 \leq p \leq \infty$ has been studied in [13], where it is showed that for any $d \in \mathbb{N}$ we have $\mathrm{Psd}([-1,1]^n) = \overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\|\cdot\|_p}$. In this same work also the closure of $\sum \mathbb{R}[\underline{X}]^{2d}$ w.r.t. weighted versions of $\|\cdot\|_p$ has been considered. In particular, Lasserre in [33] identified a weighted version $\|\cdot\|_w$ of the norm $\|\cdot\|_1$ such that for any $S \subseteq \mathbb{R}[\underline{X}]$ finite $\mathrm{Psd}(K_S) = \overline{M_S}^{\|\cdot\|_w}$.

The question of establishing when the closure of an even power module M in $\mathbb{R}[\underline{X}]$ coincides with $\mathrm{Psd}(K)$ for some subset K of $\mathbb{R}^n$ can be clearly considered also for even power modules in any unital commutative topological $\mathbb{R}-$algebra. Such a general setting was studied in [14] and [16]. We would like to present here the main result [16] as it is a powerful application of the Representation Theorem 1.3.24 and allows to deduce several of the closure results mentioned above.

Let A be a unital commutative $\mathbb{R}-$algebra and denote by $X(A)$ the character space of A (see Section 1.3.2 for the definition). For any $M \subseteq A$, recall that $\mathcal{K}_M := \{\alpha \in X(A) : \hat{a}(\alpha) \geq 0, \forall a \in M\}$, where $\hat{a}$ is the Gelfand transform of a (see Section 1.3.2 for the definition).

Definition 1.3.43. *A function $\rho : A \to \mathbb{R}$ is called a* seminorm *if*

1. ρ is subadditive: $\forall x, y \in A$, $\rho(x+y) \leq \rho(x) + \rho(y)$.

2. ρ is positively homogeneous: $\forall x \in A$, $\forall \lambda \in \mathbb{R}, \rho(\lambda x) = |\lambda|\rho(x)$.

A seminorm on a A is said to be submultiplicative *if*

$$\forall x, y \in A,\ \rho(xy) \leq \rho(x)\rho(y).$$

If ρ is a submultiplicative seminorm on A, then (A, ρ) is called a *seminormed algebra*. (In particular, A with a submultiplicative norm is said to be a normed algebra). Note that any seminormed algebra is a topological algebra with jointly continuous multiplication (c.f. [22, Proposition 1.2.14]). We denote by $\mathfrak{sp}(\rho)$ the set of all $\rho-$continuous $\mathbb{R}-$algebra homomorphisms from A to $\mathbb{R}$ and we refer to $\mathfrak{sp}(\rho)$ as the *Gelfand spectrum* of (A, ρ), i.e.

$$\mathfrak{sp}(\rho) := \{\alpha \in \underline{X}(A) : \alpha \text{ is } \rho - \text{continuous}\}.$$

Lemma 1.3.44.
For any seminormed $\mathbb{R}-$algebra (A, ρ) we have:

$$\mathfrak{sp}(\rho) = \{\alpha \in \underline{X}(A) : |\alpha(a)| \leq \rho(a) \textit{ for all } a \in A\}.$$

Proof. The inclusion $\{\alpha \in \underline{X}(A) : |\alpha(a)| \leq \rho(a) \text{ for all } a \in A\} \subseteq \mathfrak{sp}(\rho)$ follows straightforward from the definition of Gelfand spectrum of (A, ρ). Let us prove by contradiction the converse inclusion.

Suppose that $\alpha \in \underline{X}(A)$ is $\rho-$continuous but that there exists $x \in A$ s.t. $|\alpha(x)| > \rho(x)$. Take $\delta \in \mathbb{R}^+$ s.t. $|\alpha(x)| > \delta > \rho(x)$ and set $y := \frac{x}{\delta}$. Then we have $\rho(y) < 1$ and $|\alpha(y)| > 1$, which imply that $\rho(y^n) \to 0$ and $|\alpha(y^n)| \to \infty$ as $n \to \infty$, contradicting the $\rho-$continuity of α. □

We are ready now to state the main result of [16].

Theorem 1.3.45. *Let (A, ρ) be a unital commutative seminormed $\mathbb{R}-$algebra and $d \in \mathbb{N}$. If M is a $2d-$power module of A, then $\overline{M}^{\rho} = \mathrm{Psd}(\mathcal{K}_M \cap \mathfrak{sp}(\rho))$.*

In order to prove this result, let us recall some fundamental properties of unital commutative seminormed $\mathbb{R}$-algebras.

Remark 1.3.46. *Any seminormed algebra (A, ρ) can be topologically embedded into a Banach algebra $(\tilde{A}, \tilde{\rho})$, i.e. there exists $\iota : (A, \rho) \to (\tilde{A}, \tilde{\rho})$ continuous embedding (see [22, Corollary 3.3.21]). Hence, A and $\iota(A)$ are homeomorphic. Recall that a Banach algebra is a normed algebra whose underlying space is a complete normed space.*

Lemma 1.3.47. *For any unital commutative Banach $\mathbb{R}-$algebra (B, σ), any $b \in B$ and $r \in \mathbb{R}$ such that $r > \sigma(b)$, and any $k \in \mathbb{N}$, there exists $p \in B$ such that $p^k = r + b$.*

Proof. The standard power series expansion

$$(r+x)^{\frac{1}{k}} = r^{\frac{1}{k}} \left(1 + \frac{x}{r}\right)^{\frac{1}{k}} = r^{\frac{1}{k}} \sum_{j=0}^{\infty} \frac{\frac{1}{k}\left(\frac{1}{k}-1\right)\cdots\left(\frac{1}{k}-j\right)}{j!} \left(\frac{x}{r}\right)^j$$

converges absolutely for $|x| < r$. This together with the fact that (B, σ) is a Banach algebra implies that, for any $b \in B$ and any $r \in \mathbb{R}$ such that $r > \sigma(b)$, we have

$$p := r^{\frac{1}{k}} \sum_{j=0}^{\infty} \frac{\frac{1}{k}\left(\frac{1}{k}-1\right)\cdots\left(\frac{1}{k}-j\right)}{j!} \left(\frac{b}{r}\right)^j \in B$$

and $p^k = (r + b)$. □

Lemma 1.3.48. *Let (B, σ) be a unital Banach $\mathbb{R}$-algebra and $L : B \to \mathbb{R}$ a linear functional. If there exists $d \in \mathbb{N}$ such that $L(b^{2d}) \geq 0$ for all $b \in B$, then L is $\sigma-$continuous.*

Proof. By Lemma 1.3.47, for all $n \in \mathbb{N}$ and all $a \in B$ we have that $\frac{1}{n} + \sigma(a) \pm a = 1 + \sigma(\pm a) + (\pm a) \in B^{2d}$. Applying L, we obtain $|L(a)| \leq (\frac{1}{n} + \sigma(a))L(1)$ for all $n \in \mathbb{N}$ and all $a \in B$, so $|L(a)| \leq \sigma(a)L(1)$ for all $a \in B$. Hence, L is $\sigma-$continuous. □

We are finally ready to show Theorem 1.3.45.

Proof. of Theorem 1.3.45
Since

$$\mathrm{Psd}(\mathcal{K}_M \cap \mathfrak{sp}(\rho)) = \bigcap_{\alpha \in \mathcal{K}_M \cap \mathfrak{sp}(\rho)} \alpha^{-1}([0, +\infty))$$

and any $\alpha \in \mathcal{K}_M \cap \mathfrak{sp}(\rho)$ is $\rho-$continuous, we have that $\mathrm{Psd}(\mathcal{K}_M \cap \mathfrak{sp}(\rho))$ is closed in (A, ρ). Hence, $\overline{M}^{\rho} \subseteq \overline{\mathrm{Psd}(\mathcal{K}_M \cap \mathfrak{sp}(\rho))}^{\rho} = \mathrm{Psd}(\mathcal{K}_M \cap \mathfrak{sp}(\rho))$. For the reverse inclusion, let us consider $b \in \operatorname{Psd}(\mathcal{K}_M \cap \mathfrak{sp}(\rho))$ and denote by $\widetilde{M}$ the closure of the image of M in the Banach algebra $(\tilde{A}, \tilde{\rho})$, i.e. $\widetilde{M} := \overline{\iota(M)}^{\tilde{\rho}}$ (see Remark 1.3.46). Then $\widetilde{M}$ is a $2d-$power module of $\tilde{A}$ as addition and multiplication on $\tilde{A}$ are both $\tilde{\rho}$-continuous and M is a $2d-$power module of A. By Lemma 1.3.47, for any $n \in \mathbb{N}$ and all $a \in \tilde{A}$ we have $\frac{1}{n}+\tilde{\rho}(a) \pm a \in \tilde{A}^{2d} \subseteq \widetilde{M}$. Hence, $\tilde{\rho}(a) \pm a \in \widetilde{M}$ which implies that $\widetilde{M}$ is Archimedean. Now, for any $\alpha \in \mathcal{K}_{\widetilde{M}}$ we have that $\alpha(a) \geq 0$ for all $a \in \widetilde{M}$, which gives in particular that α is a linear functional on $\tilde{A}$ s.t. $\alpha(a^{2d}) \geq 0$ for all $a \in \tilde{A}$ and so Lemma 1.3.48 ensures that α is $\tilde{\rho}-$continuous. Hence, $\alpha \circ \iota$ is $\rho-$continuous and $\alpha(\iota(m)) \geq 0$ for all $m \in M$, i.e.

$$(\alpha \circ \iota) \in \mathcal{K}_M \cap \mathfrak{sp}(\rho), \forall \alpha \in \mathcal{K}_{\widetilde{M}}. \tag{1.5}$$

Denote by $\tilde{b} := \iota(b)$. Then (1.5) ensures that for all $\alpha \in \mathcal{K}_{\widetilde{M}}$ we have $\alpha(\tilde{b}) = (\alpha \circ \iota)(b) \geq 0$ as by assumption $b \in \operatorname{Psd}(\mathcal{K}_M \cap \mathfrak{sp}(\rho))$. By Jacobi-Prestel Nichnegativstellensatz we have that for all $n \in \mathbb{N}$, $\tilde{b} + \frac{1}{n} \in \widetilde{M}$ and so by the completeness of $\tilde{A}$ we get $\tilde{b} \in \widetilde{M}$. This yields $\iota(b) \in \overline{\iota(M)}^{\tilde{\rho}} = \iota(\overline{M}^{\rho})$ where the latter equality holds since A and $\iota(A)$ are homeomorphic (see Remark 1.3.46). Hence, $b \in \overline{M}^{\rho}$. □

Keeping in mind the identification between $X(\mathbb{R}[\underline{X}])$ and $\mathbb{R}^n$ proved in Proposition 1.3.26 and applying Theorem 1.3.45 for $A = \mathbb{R}[\underline{X}]$, we obtain some of the closure results mentioned above.

Examples 1.3.49. *Let $M := \sum \mathbb{R}[\underline{X}]^2$ and so $K_M = \mathbb{R}^n$.*

(a) If we consider the norm $\|\cdot\|_1$ defined by $\|f\|_1 := \sum_\beta |f_\beta|$ for all $f = \sum_\beta f_\beta \underline{X}^\beta \in \mathbb{R}[\underline{X}]$, then $(\mathbb{R}[\underline{X}], \|\cdot\|_1)$ is a normed algebra. Hence, Theorem 1.3.45 gives $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_1} = \operatorname{Psd}(\mathbb{R}^n \cap \mathfrak{sp}(\|\cdot\|_1))$. Let us now compute the Gelfand spectrum of $(\mathbb{R}[\underline{X}], \|\cdot\|_1)$.
If $y = (y_1, \ldots, y_n) \in \mathfrak{sp}(\|\cdot\|_1)$, then by Lemma 2.3.8 we obtain that $|p(y)| \leq \|p\|_1$ for all $p \in \mathbb{R}[\underline{X}]$ and in particular for each $i = 1, \ldots, n$ we have $|y_i| \leq \|X_i\|_1 = 1$. Hence, $y \in [-1,1]^n$. Conversely, for any $y = (y_1, \ldots, y_n) \in [-1,1]^n$ we have that $|y_i| = 1$ for $i = 1, \ldots, n$ and so for any $p = \sum_\beta p_\beta \underline{X}^\beta \in \mathbb{R}[\underline{X}]$ we get

$$|p(y)| \leq \sum_\beta |p_\beta||y_1|^{\beta_1} \cdot |y_n|^{\beta_n} \leq \sum_\beta |p_\beta| = \|p\|_1.$$

Hence, by Lemma 2.3.8, $y \in \mathfrak{sp}(\|\cdot\|_1)$.

We have therefore showed that $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_1} = \operatorname{Psd}([-1,1]^n)$, retrieving the result of [2] and [34].

(b) Let $1 \leq p < \infty$ *and consider* $\|\cdot\|_p$*, where for any* $f = \sum_\beta f_\beta \underline{X}^\beta \in \mathbb{R}[\underline{X}]$ *we define* $\|f\|_p := \left(\sum_\beta |f_\beta|^p\right)^{\frac{1}{p}}$ *for* $1 \leq p < \infty$ *and* $\|f\|_\infty := \max_\beta |f_\beta|$*. As* $\|f\|_p \leq \|f\|_1$ *for all* $f \in \mathbb{R}[\underline{X}]$*, we have that* $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_1} \subseteq \overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_p}$ *and so by (a) we obtain* $\mathrm{Psd}([-1,1]^n) \subseteq \overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_p}$*. Furthermore, for any* $y \in [-1,1]^n$ *we have that the map* $e_y : \mathbb{R}[\underline{X}] \to \mathbb{R}$*, defined by* $e_y(f) := f(y)$ *for any* $f \in \mathbb{R}[\underline{X}]$*, is* $\|\cdot\|_p-$*continuous. Indeed, for any* $y = (y_1, \ldots, y_n) \in [-1,1]^n$ *and any* $f = \sum_\beta f_\beta \underline{X}^\beta \in \mathbb{R}[\underline{X}]$ *we have that*

$$|e_y(f)| = |f(y)| \leq \sum_\beta |f_\beta||y_1|^{\beta_1} \cdots |y_n|^{\beta_n} \overset{\text{Hölder ineq.}}{\leq} C_q \|f\|_p,$$

where $1 \leq q \leq \infty$ *is such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and*

$$C_q := \begin{cases} \left(\sum_\beta |y_1|^{q\beta_1} \cdots |y_n|^{q\beta_n}\right)^{\frac{1}{q}} & \text{if } q < \infty \\ \max_\beta |y_1|^{\beta_1} \cdots |y_n|^{\beta_n} & \text{if } q = \infty \end{cases}$$

which is finite as $y = (y_1, \ldots, y_n) \in [-1,1]^n$*.*

Hence, $\mathrm{Psd}([-1,1]^n) = \bigcap_{y \in [-1,1]^n} e_y^{-1}([0,+\infty))$ *is closed in* $(\mathbb{R}[\underline{X}], \|\cdot\|_p)$*, which yields* $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_p} \subseteq \overline{\mathrm{Psd}([-1,1]^n)}^{\|\cdot\|_p} = \mathrm{Psd}([-1,1]^n)$*. We have therefore showed that* $\mathrm{Psd}([-1,1]^n) = \overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_p}$*, for all* $1 \leq p \leq \infty$*, retrieving the result of [13].*

Theorem 1.3.45 easily extends to locally multiplicatively convex algebras.

Definition 1.3.50. *A unital commutative* $\mathbb{R}-$*algebra* A *endowed with a locally convex topology induced by a family of submultiplicative seminorms on* A *is called* locally multiplicatively convex (lmc)*.*

If (A, τ) is an lmc algebra, then it is a topological algebra with jointly continuous multiplication (c.f. [22, Proposition 2.1.9]). Moreover, we denote by $\mathfrak{sp}(\tau)$ the set of all $\tau-$continuous $\mathbb{R}-$algebra homomorphisms from A to $\mathbb{R}$ and we refer to $\mathfrak{sp}(\tau)$ as the *Gelfand spectrum* of (A, τ).

Using that any locally convex topology can be always generated by a family of directed seminorms (see [21, Proposition 4.2.14]) we get the following result.

Proposition 1.3.51. *Let* (A, τ) *be an lmc algebra with* τ *generated by a directed family* $\mathcal{F}$ *of submultiplicative seminorms. Then* $\mathfrak{sp}(\tau) = \bigcup_{\rho \in \mathcal{F}} \mathfrak{sp}(\rho)$*.*

Proof. Applying [21, Proposition 4.6.1] and the definition of Gelfand spectrum, we easily obtain

$$\begin{aligned}\mathfrak{sp}(\tau) &= \{\alpha \in X(A) : \alpha \text{ is } \tau-\text{continuous}\} \\ &= \bigcup_{\rho\in\mathcal{F}} \{\alpha \in X(A) : \alpha \text{ is } \rho-\text{continuous}\} = \bigcup_{\rho\in\mathcal{F}} \mathfrak{sp}(\rho).\end{aligned}$$

□

It is then clear how to extend Theorem 1.3.45 to any lmc algebra.

Theorem 1.3.52. *Let (A, τ) be an lmc algebra and $d \in \mathbb{N}$. If M is a $2d-$power module of A, then $\overline{M}^{\tau} = \mathrm{Psd}(\mathcal{K}_M \cap \mathfrak{sp}(\tau))$.*

Proof. Let $\mathcal{F}$ be a directed family of submultiplicative seminorms generating τ. Then by Proposition 1.3.51, we get

$$\begin{aligned}\overline{M}^{\tau} &= \bigcap_{\rho\in\mathcal{F}} \overline{M}^{\rho} = \bigcap_{\rho\in\mathcal{F}} \mathrm{Psd}\left(\mathcal{K}_M \cap \mathfrak{sp}(\rho)\right) \\ &= \mathrm{Psd}\left(\mathcal{K}_M \cap \bigcup_{\rho\in\mathcal{F}} \mathfrak{sp}(\rho)\right) = \mathrm{Psd}\left(\mathcal{K}_M \cap \mathfrak{sp}(\tau)\right).\end{aligned}$$

□

Chapter 2

$K-$Moment Problem: formulation and connection to $\mathrm{Psd}(K)$

2.1 Formulation: from finite to infinite dimensional settings.

As suggested by the name the $K-$Moment Problem deals with moments of measures. In this course we are going to consider always *non-negative Radon measures* on Hausdorff topological spaces.

Recall that

Definition 2.1.1. *A* Radon measure μ *on a Hausdorff space* (X, τ) *is a measure defined on the Borel* $\sigma-$*algebra* $\mathcal{B}_\tau$ *on* (X, τ) *(i.e. the smallest* $\sigma-$*algebra on* X *containing* τ*) and such that*

- μ *is locally finite, i.e. for all* $x \in X$ *there exists* U *open neighbourhood of* x *in* (X, τ) *such that* $\mu(U) < \infty$*)*
- μ *is inner regular, i.e. for all* $B \in \mathcal{B}_\tau$, $\mu(B) = \sup\{\mu(K) : K \subseteq B \, compact\}$.

We say that μ *is* supported in *a subset* Y *of* X *if for any* $B \in \mathcal{B}_\tau$ *we have that* $B \cap Y = \emptyset$ *implies* $\mu(B) = 0$.

Let us start by introducing the most classical version of the $K-$moment problem.

Given a Radon measure μ on $\mathbb{R}$ and $j \in \mathbb{N}_0$, the $j-th$ *moment of* μ is defined as

$$m_j^\mu := \int_{\mathbb{R}} x^j \mu(dx).$$

If all moments of μ exist and are finite, then we can associate to μ the sequence of real numbers $(m_j^\mu)_{j \in \mathbb{N}_0}$, which is said to be the *moment sequence* of μ. The moment problem exactly addresses the inverse question:

Problem 2.1.2 (The one-dimensional $K-$Moment Problem (KMP))**.**
Let $N \in \mathbb{N}_0 \cup +\infty$. Given a closed subset K of $\mathbb{R}$ and a sequence $m := (m_j)_{j=0}^N$ of real numbers, does there exist a non-negative Radon measure μ supported in K and s.t. $m_j = m_j^\mu$ for all $j = 0, 1, \ldots, N$, i.e.

$$m_j = \int_K x^j \mu(dx), \ \forall\, j = 0, 1, \ldots, N?$$

If such a measure μ does exist we say that μ is a $K-$*representing measure* for m or that m is represented by μ on K. If $N = \infty$ the KMP is said to be *full*, while it is called *truncated* if $N < \infty$. In the following we are going to focus on the full KMP.

Note that there is a bijective correspondence between the set $\mathbb{R}^{\mathbb{N}_0}$ of all sequences of real numbers and the set $(\mathbb{R}[X])^*$ of all linear functionals on $\mathbb{R}[X]$, namely

$$\begin{array}{llllll} \phi: & \mathbb{R}^{\mathbb{N}_0} & \to & (\mathbb{R}[X])^* & & \\ & m := (m_j)_{j\in\mathbb{N}_0} & \mapsto & L_m: \ \mathbb{R}[x] & \to & \mathbb{R} \\ & & & \quad p := \sum\limits_j p_j X^j & \mapsto & L_m(p) := \sum\limits_j p_j m_j, \end{array}$$

where L_m is called *Riesz' functional*. Indeed

- ϕ is injective, because if $m := (m_j)_{j\in\mathbb{N}_0}, m' := (m'_j)_{j\in\mathbb{N}_0} \in \mathbb{R}^{\mathbb{N}_0}$ and $m \neq m'$ then there exists $j \in \mathbb{N}_0$ s.t. $m_j \neq m'_j$, i.e. $L_m(x^j) \neq L_{m'}(x^j)$, and so $\phi(m) = L_m \neq L_{m'} = \phi(m')$.
- ϕ is surjective, because for any $\ell \in (\mathbb{R}[X])^*$ the sequence $m := (\ell(X^j))_{j\in\mathbb{N}_0}$ is such that $\phi(m) = \ell$. In fact, for any $p := \sum\limits_j p_j X^j \in \mathbb{R}[X]$ we have
$$L_m(p) = \sum_j p_j \ell(X^j) = \ell\left(\sum_j p_j X^j\right) = \ell(p) \text{ and, hence, } \phi(m) = L_m = \ell.$$

In virtue of this correspondence, we can always reformulate the full KMP in terms of linear functionals.

Problem 2.1.3 (The one-dimensional $K-$Moment Problem (KMP))**.**
Given a closed subset K of $\mathbb{R}$ and $L : \mathbb{R}[X] \to \mathbb{R}$ linear, does there exists a non-negative Radon measure μ supported in K s.t. $L(p) = \int p d\mu$, $\forall p \in \mathbb{R}[X]$?

If such a measure exists we say that μ is a $K-$*representing measure* for L and that it is a solution to the $K-$moment problem for L.

This reformulation makes clearly how to generalize the statement of the one-dimensional KMP to higher dimensions (see also [21, Section 5.2.2]). Let $n \in \mathbb{N}$ and $\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$.

Problem 2.1.4 (The n-dimensional $K-$Moment Problem (KMP))**.**
Given a closed subset K of $\mathbb{R}^n$ and $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ linear, does there exists a non-negative Radon measure μ supported in K s.t. $L(p) = \int p d\mu$, $\forall p \in \mathbb{R}[\underline{X}]$?

We can clearly consider also infinite dimensional settings, e.g. by replacing $\mathbb{R}[X_1, \dots, X_n]$ with $\mathbb{R}[X_i : i \in \Omega]$, where Ω is an infinite index set or replacing the polynomial algebra by any infinitely generated unital commutative $\mathbb{R}-$algebra. Let us then give a formulation of the $K-$moment problem general enough to encompass all the above mentioned instances.

Given a unital commutative $\mathbb{R}-$algebra A, recall that we denote by $X(A)$ its character space of A (see Section 1.3.2). We endow the character space $X(A)$ with the weakest topology $\tau_{X(A)}$ on $X(A)$ s.t. all Gelfand transforms are continuous, i.e. $\hat{a} : X(A) \to \mathbb{R}$, $\hat{a}(\alpha) := \alpha(a)$ is continuous for all $a \in A$. A basis for $\tau_{X(A)}$ is given by

$$\mathcal{N} := \left\{ \bigcap_{i=1}^{n} \hat{a_i}^{-1}(U_{a_i}) : a_1, \dots, a_n \in A, U_{a_1}, \dots, U_{a_n} \text{open in } \mathbb{R}, n \in \mathbb{N} \right\}.$$

Remark 2.1.5. *$X(A)$ can be seen as a subset of $\mathbb{R}^A$ via the embedding:*

$$\begin{array}{rccl} \pi : & X(A) & \to & \mathbb{R}^A \\ & \alpha & \mapsto & \pi(\alpha) := \left(\alpha(a)\right)_{a \in A} = \left(\hat{a}(\alpha)\right)_{a \in A}. \end{array}$$

If we equip $\mathbb{R}^A$ with the product topology τ_{prod}, then $\tau_{X(A)}$ coincides with the topology τ_π induced by π on $X(A)$ from $(\mathbb{R}^A, \tau_{prod})$, i.e.

$$\tau_{X(A)} \equiv \left\{ \pi^{-1}(O) : O \in \tau_{prod} \right\}.$$

Hence, π is a topological embedding and the space $\left(X(A), \tau_{X(A)}\right)$ is Hausdorff.

Proof. Let $a \in A$. Then π is $\tau_\pi-$continuous and the projection $p_a : \mathbb{R}^A \to \mathbb{R}$, $p_a((x_b)_{b \in A}) := x_a$ is $\tau_{prod}-$continuous. Hence, $\hat{a} = p_a \circ \pi$ is $\tau_\pi-$continuous and so $\tau_{X(A)} \subseteq \tau_\pi$.

Conversely, let $O \in \tau_{prod}$. Then there exist $n \in \mathbb{N}$, $b_1, \dots, b_n \in A$ and $U_{b_1}, \dots, U_{b_n}$ open in $\mathbb{R}$ such that $\prod_{i=1}^{n} U_{b_i} \times \prod_{a \in A \setminus \{b_1, \dots, b_n\}} \mathbb{R} \subseteq O$. Hence, $\pi^{-1}(O) \supseteq \pi^{-1}\left(\bigcap_{i=1}^n p_{b_i}^{-1}(U_{b_i})\right) = \bigcap_{i=1}^n \pi^{-1}\left(p_{b_i}^{-1}(U_{b_i})\right) = \bigcap_{i=1}^n \hat{b_i}^{-1}(U_{b_i}) \in \mathcal{N}$ and so $\tau_\pi \subseteq \tau_{X(A)}$ □

We are now ready to introduce the general formulation of KMP announced above.

Problem 2.1.6 (The KMP for unital commutative $\mathbb{R}-$algebras)**.**
Let A be a unital commutative $\mathbb{R}-$algebra. Given a closed subset $K \subseteq X(A)$ and $L : A \to \mathbb{R}$ linear, does there exist a non-negative Radon measure μ on $X(A)$ supported on K and such that $L(a) = \int_{X(A)} \hat{a}(\alpha)\mu(d\alpha), \forall a \in A$?

Note that for $A = \mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \ldots, X_n]$ Problem 2.1.6 reduces to Problem 2.1.4 by means of the correspondence $X(\mathbb{R}[\underline{X}]) \cong \mathbb{R}^n$ introduced in Proposition 1.3.26.

2.2 Riesz-Haviland's Theorem

Let A be a unital commutative $\mathbb{R}-$algebra. Given a subset K of $X(A)$, we denote by

$$\mathrm{Psd}(K) := \{a \in A : \hat{a} \geq 0 \text{ on } K\}.$$

A necessary condition for the existence of a solution to Problem 2.1.6 is clearly that L is nonnegative on $\mathrm{Psd}(K)$. In fact, if there exists a $K-$representing measure μ for L then for all $a \in \mathrm{Psd}(K)$ we have

$$L(a) = \int_{X(A)} \hat{a}(\alpha)\mu(d\alpha) \geq 0$$

since μ is nonnegative and supported on K and $\hat{a}$ is nonnegative on K.

It is then natural to ask if the non-negativity of L on $\mathrm{Psd}(K)$ is also sufficient. For $A = \mathbb{R}[X_1, \ldots, X_n]$ a positive answer is provided by the so-called Riesz-Haviland theorem (see [46, 20]).

Theorem 2.2.1 (Classical Riesz-Haviland Theorem)**.** *Let $K \subseteq \mathbb{R}^n$ closed and $L : \mathbb{R}[X_1, \ldots, X_n] \to \mathbb{R}$ linear. Then L has a $K-$representing measure if and only if $L(\mathrm{Psd}(K)) \subseteq [0, +\infty)$.*

An analogous result also holds in the general setting.

Theorem 2.2.2 (Generalized Riesz-Haviland Theorem)**.** *Let $K \subseteq X(A)$ closed and $L : A \to \mathbb{R}$ linear. Suppose there exists $p \in A$ such that $\hat{p} \geq 0$ on K and for all $n \in \mathbb{N}$ the set $\{\alpha \in K : \hat{p}(\alpha) \leq n\}$ is compact. Then L has a $K-$representing measure if and only if $L(\mathrm{Psd}(K)) \subseteq [0, +\infty)$.*

This theorem reduces the solvability of the $K-$moment problem to the problem of characterizing $\mathrm{Psd}(K)$ establishing the beautiful duality between these two problems.

We will prove both Theorems 2.2.1 and 2.2.2 as corollaries of the following more general result for which we need some notation. Given a topological space (X, τ), we denote by $\mathcal{C}(X)$ the space of all continuous real valued functions defined and by $\mathcal{C}_c(X)$ the subspace of all functions in $\mathcal{C}(X)$ having compact support $\text{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}^{\tau}$.

Theorem 2.2.3. *Let A be a unital commutative $\mathbb{R}-$algebra, χ a Hausdorff space and $\hat{}: A \to \mathcal{C}(\chi)$ a $\mathbb{R}-$algebra homomorphism. Suppose that*

$$\exists\, p \in A \text{ s.t. } \hat{p} \geq 0 \text{ on } \chi \text{ and } \forall j \in \mathbb{N},\ \chi_j := \{\alpha \in \chi : \hat{p}(\alpha) \leq j\} \text{ is compact.} \tag{2.1}$$

If $L : A \to \mathbb{R}$ is linear and s.t. $L(a) \geq 0$ for all $a \in A$ with $\hat{a} \geq 0$ on χ, then there exists a Radon measure μ on χ such that $L(a) = \int \hat{a} d\mu$, for all $a \in A$.

Remark 2.2.4. (2.1) *implies that χ is locally compact, i.e. for any $x \in \chi$ there exists a compact neighbourhood of x.*

Proof.
Let $x \in \chi$ and $j \in \mathbb{N}$ such that $\hat{p}(x) < j$. Then $U := \{y \in \chi \mid \hat{p}(y) < j\} \subseteq \chi_j$, $x \in U$, and U is open (since $U = \hat{p}^{-1}\big((-\infty, j)\big)$ and $\hat{p} \in \mathcal{C}(\chi)$). Hence, U is an open neighbourhood of x and so $\overline{U}$ is a closed neighbourhood of x contained in χ_j, which is compact. Then, $\overline{U}$ is a compact neighbourhood of x. □

Proof. of Theorem 2.2.1
Let $\chi := K$ be a closed subset of $\mathbb{R}^n$, $A := \mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$, $\hat{} :$ $\mathbb{R}[X_1, \ldots, X_n] \to \mathcal{C}(K)$ defined by $\hat{f} := f \restriction_K$, and $p := \sum_{i=1}^n X_i^2$ i.e. $p = \|\underline{X}\|^2$, where $\|\cdot\|$ is the euclidean norm on $\mathbb{R}^n$. Then $\hat{p} \geq 0$ on K and for any $j \in \mathbb{N}$ the $\chi_j = \{x \in K : \|x\|^2 \leq j\}$ is compact. Hence, (2.1) holds and the conclusion follows by Theorem 2.2.3. □

Proof. of Theorem 2.2.2
Let $\chi := K$ be a closed subset of $X(A)$ endowed with the subset topology induced by $\tau_{X(A)}$ which makes K into a Hausdorff space. Define the map

$$\begin{array}{rcl} \hat{}: A & \to & \mathcal{C}(K) \\ a & \mapsto & \hat{a} \restriction_K, \end{array}$$

where $\hat{a}$ is the Gelfand transform of a. This is well-defined as the Gelfand transform of a restricted to K is a continuous $\mathbb{R}-$algebra homomorphism. Then the conclusion follows by Theorem 2.2.3. □

Theorem 2.2.3 was most probably known since at least the sixties as it can be derived from a theorem due to Choquet in [7]. However, we propose a proof due to Marshall, see [37, Theorem 3.2.2] or [36, Theorem 3.1], and based on the following famous result.

Theorem 2.2.5 (Riesz-Markov-Kakutani theorem)**.** *Let χ be a locally compact Hausdorff space. If $L : \mathcal{C}_c(\chi) \to \mathbb{R}$ is a positive linear functional, i.e. $L(f) \geq 0$ for all $f \in \mathcal{C}_c(\chi)$ such that $f \geq 0$ on χ, then there exists a unique non-negative Borel regular measure μ on χ such that $L(f) = \int f d\mu$ for all $f \in \mathcal{C}_c(\chi)$.*

Proof. (see e.g. [28, Theorem 16, p.77]) □

Recall that a Borel regular measure μ on the Hausdorff space (χ, τ) is a measure defined on the Borel $\sigma-$algebra $\mathcal{B}_\tau$ such that μ is both inner regular and outer regular, where μ outer regular means that for all $B \in \mathcal{B}_\tau$, $\mu(B) = \inf\{\mu(O) : O \supseteq B \text{open}\}$. Note that a finite Borel regular measure is in particular a Radon measure.

Proof. of Theorem 2.2.3
Let $\hat{A} := \{\hat{a} : a \in A\}$ and $\mathcal{B}(\chi) := \{f \in \mathcal{C}(\chi) : \exists a \in A \text{ s.t.} |f| \leq |\hat{a}| \text{ on } \chi\}$. Since $\hat{} : A \to \mathcal{C}(\chi)$ is an $\mathbb{R}-$algebra homomorphism, we have that both $\hat{A}$ and $\mathcal{B}(\chi)$ are subalgebras of $\mathcal{C}(\chi)$ and $\hat{A} \subseteq \mathcal{B}(\chi) \subseteq \mathcal{C}(\chi)$.

Claim 1: $\mathcal{C}_c(\chi)$ is a subalgebra of $\mathcal{B}(\chi)$.

Proof of Claim 1.
Clearly, $\mathcal{C}_c(\chi)$ equipped with the pointwise operations of addition and multiplication is an $\mathbb{R}-$algebra. Moreover, if $f \in \mathcal{C}_c(\chi)$ then f is bounded above on χ, and so there exists $k \in \mathbb{N}$ s.t. $|f| \leq k$ on χ. Since $k \in A$, we have that $|f| \leq \hat{k}$ on χ, i.e. $f \in \mathcal{B}(\chi)$. Hence, $\mathcal{C}_c(\chi)$ is a subalgebra of $\mathcal{B}(\chi)$. □(Claim 1)

Define $\overline{L} : \hat{A} \to \mathbb{R}$ as $\overline{L}(\hat{a}) = L(a)$ for all $a \in A$.

Claim 2: $\overline{L}$ is a well-defined linear functional on $\hat{A}$.

Proof of Claim 2.
It is enough to prove that

$$\forall a \in A,\ \hat{a} = 0 \Rightarrow L(a) = 0. \tag{2.2}$$

In fact, (2.2) implies that $\overline{L}(a) = \overline{L}(b)$ for any $a, b \in A$ such that $\hat{a} = \hat{b}$, i.e. $\overline{L}$ is well-defined. Also, using (2.2) together with the assumptions that $\hat{}$ is a

$\mathbb{R}-$algebra homomorphism and L is linear, we obtain that for any $a, b \in A$ and $\lambda \in \mathbb{R}$

$$\overline{L}(\hat{a}+\hat{b}) \overset{(2.2)}{=} \overline{L}(\widehat{a+b}) = L(a+b) = L(a)+L(b) = \overline{L}(\hat{a})+\overline{L}(\hat{b})$$

and

$$\overline{L}(\lambda\hat{a}) \overset{(2.2)}{=} \overline{L}(\widehat{\lambda a}) = L(\lambda a) = \lambda L(a) = \lambda\overline{L}(\hat{a}).$$

Let us then show that (2.2) holds. If $\hat{a} = 0$ then $\hat{a} \geq 0$ and $-\hat{a} = \widehat{-a} \geq 0$. These respectively imply that $L(a) \geq 0$ and $L(-a) \geq 0$, which together yield $L(a) = 0$, i.e. $\overline{L}(\hat{a}) = 0$.

□(Claim 2)

Claim 3: $\overline{L} : \hat{A} \to \mathbb{R}$ extends to a linear functional $\overline{\overline{L}} : \mathcal{B}(\chi) \to \mathbb{R}$ s.t. $\overline{\overline{L}}(f) \geq 0$ for all $f \in \mathcal{B}(\chi)$ with $f \geq 0$ on χ.

Proof of Claim 3.

Consider the collection $\mathcal{P}$ of all pairs $\left(V, \overline{\overline{L}}\right)$, where V is a $\mathbb{R}-$subspace of $\mathcal{B}(\chi)$ containing $\hat{A}$ and $\overline{\overline{L}}$ is an extension of $\overline{L} : \hat{A} \to \mathbb{R}$ such that $\overline{\overline{L}}(f) \geq 0$ for all $f \in V$ with $f \geq 0$ on χ. Define the following partial order on $\mathcal{P}$

$$\left(V_1, \overline{\overline{L}}_1\right) \subseteq \left(V_2, \overline{\overline{L}}_2\right) \Longleftrightarrow V_1 \subseteq V_2 \text{ and } \overline{\overline{L}}_2 \restriction_{V_1} = \overline{\overline{L}}_1.$$

- $\mathcal{P}$ is non-empty since $\left(\hat{A}, \overline{L}\right)$ belongs to it. In fact, for any $a \in A$ s.t. $\hat{a} \geq 0$ on χ we have $\overline{L}(\hat{a}) = L(a) \geq 0$, where the latter inequality holds by assumption on L.
- Every chain in $\mathcal{P}$ has an upper bound. Indeed, for any $\{(V_i, \ell_i), i \in J\}$ chain in $\mathcal{P}$, the pair $\left(\bigcup_{i\in J} V_i, \ell\right)$ is an upper bound, where the functional $\ell : \bigcup_{i\in J} V_i \to \mathbb{R}$ is linear and such that $\ell \restriction_{V_i} = \ell_i$ for all $i \in J$.

Then by Zorn's lemma there exists be a maximal element $\left(B, \overline{\overline{L}}\right)$ in $\mathcal{P}$.
We want to show that $B = \mathcal{B}(\chi)$.

Suppose that this is not the case and let $g \in \mathcal{B}(\chi) \setminus B$. If $f_1, f_2 \in B$ s.t. $f_1 \leq g$ and $g \leq f_2$ on χ, then $f_1 \leq f_2$ on χ, and so $\overline{\overline{L}}(f_1) \leq \overline{\overline{L}}(f_2)$. Therefore,

$$\mathcal{U} := \{\overline{\overline{L}}(f_1) : \ f_1 \in B, f_1 \leq g \text{ on } \chi\} \text{ and } \Theta := \{\overline{\overline{L}}(f_2) : \ f_2 \in B, g \leq f_2 \text{ on } \chi\}$$

are such that $u \leq \theta$ for all $u \in \mathcal{U}$ and $\theta \in \Theta$. Moreover, $\mathcal{U}$ and Θ are both non-empty. [Indeed, as $g \in \mathcal{B}(\chi)$, there exists $a \in A$ s.t. $|g| \leq |\hat{a}|$ on χ and so $|\hat{a}| \leq \dfrac{\hat{a}^2+1}{2} \in \hat{A}$ (since $\left(\hat{a} \pm 1\right)^2 \geq 0$), which in turns gives that

$f_1 := -\dfrac{\hat{a}^2+1}{2} \in \hat{A}$ and $f_2 := \dfrac{\hat{a}^2+1}{2} \in \hat{A}$ are such that $f_1 \leq g \leq f_2$.] The completeness of $\mathbb{R}$ ensures that

$$\exists\ e \in \mathbb{R} \text{ s.t. } \sup(\mathcal{U}) \leq e \leq \inf(\Theta). \tag{2.3}$$

We can now linearly extend $\overline{\overline{L}}$ from B to $B + \mathbb{R}g \subseteq \mathcal{B}(\chi)$ by setting $\overline{\overline{L}}(g) := e$ and so $\overline{\overline{L}}(f + dg) := \bar{L}(f) + de$ for all $d \in \mathbb{R}$ and $f \in B$. Then the following holds

$$\forall\ f + dg \in B + \mathbb{R}g,\ f + dg \geq 0 \text{ on } \chi \Rightarrow \overline{\overline{L}}(f + dg) \geq 0, \tag{2.4}$$

which yields $\left(B + \mathbb{R}g, \overline{\overline{L}}\right) \supseteq (B, \overline{\overline{L}})$ and so contradicts the maximality of $(B, \overline{\overline{L}})$, proving that $B = \mathcal{B}(\chi)$. To show that (2.4) holds, we need to distinguish three cases.

Case 1: If $d = 0$ and $f + dg \in B + \mathbb{R}g$ is s.t. $f + dg \geq 0$ on χ, then $\overline{\overline{L}}(f) \geq 0$ since $\left(B, \overline{\overline{L}}\right) \in \mathcal{P}$.

Case 2: If $d > 0$ and $f + dg \in B + \mathbb{R}g$ is s.t. $f + dg \geq 0$ on χ, then $-\frac{f}{d} \leq g$ on χ. Hence, $\overline{\overline{L}}\left(-\frac{f}{d}\right) \in \mathcal{U}$ and so by (2.3) we have $\overline{\overline{L}}\left(-\frac{f}{d}\right) \leq e = \overline{\overline{L}}(g)$, i.e. $0 \leq \overline{\overline{L}}(g) - \overline{\overline{L}}\left(-\frac{f}{d}\right) = \overline{\overline{L}}\left(g + \frac{f}{d}\right) = \frac{1}{d}\overline{\overline{L}}\left(f + gd\right)$. Then $\overline{\overline{L}}\left(f + gd\right) \geq 0$.

Case 3: If $d < 0$ and $f + dg \in B + \mathbb{R}g$ is s.t. $f + dg \geq 0$ on χ, then $-\frac{f}{d} \geq g$ on χ. Hence, $\overline{\overline{L}}\left(-\frac{f}{d}\right) \in \Theta$ and so by (2.3) we have $\overline{\overline{L}}\left(-\frac{f}{d}\right) \geq e = \overline{\overline{L}}(g)$, i.e. $0 \leq \overline{\overline{L}}(g) - \overline{\overline{L}}\left(-\frac{f}{d}\right) = \overline{\overline{L}}\left(g + \frac{f}{d}\right) = -\frac{1}{d}\overline{\overline{L}}\left(f + gd\right)$. Then $\overline{\overline{L}}\left(f + gd\right) \geq 0$.

□(Claim 3)

By Claim 1, we know that $\mathcal{C}_c(\chi) \subseteq \mathcal{B}(\chi)$ and so $\overline{\overline{L}}$ is in particular defined on $\mathcal{C}_c(\chi)$ and such that $\overline{\overline{L}}(f) \geq 0$ for all $f \in \mathcal{C}_c(\chi)$ with $f \geq 0$ on χ. This together with Remark 2.2.4 guarantees that we can apply Theorem 2.2.5 and, hence, that

$$\exists\ \mu \text{ Borel regular measure on } \chi \text{ s.t. } \overline{\overline{L}}(f) = \int f d\mu,\ \ \forall f \in \mathcal{C}_c(\chi). \tag{2.5}$$

Main Claim: $\overline{\overline{L}}(f) = \int f d\mu, \forall f \in \mathcal{B}(\chi)$.

Proof of Main Claim.
Let $f \in \mathcal{B}(\chi)$. W.l.o.g. we can assume that $f \geq 0$ on χ, since $f = f_+ - f_-$ where $f_+ := \max\{f, 0\}$ and $f_- := -\min\{f, 0\}$. Set $q := f + \hat{p}$ where p is the one in (2.1). Then $q \in \mathcal{B}(\chi)$.

For each $j \in \mathbb{N}$, define $\chi_j^{'} := \{x \in \chi \mid q(x) \leq j\}$. Then

- $\forall\ j \in \mathbb{N}, \chi_j^{'}$ is compact. Indeed, for all $x \in \chi$ we have that $q(x) \geq \hat{p}(x)$ and so that $\chi_j^{'} \subseteq \chi_j$, which yields that $\chi_j^{'}$ is closed subset of a compact set and so itself compact.
- $\chi_j^{'} \subseteq \chi_{j+1}^{'}$ and $\chi = \bigcup_j \chi_j^{'}$.

Subclaim 1: For each $j \in \mathbb{N}$, there exists $f_j \in \mathcal{C}_c(\chi)$ such that $0 \leq f_j \leq f$, $f_j = f$ on $\chi_j^{'}$ and $f_j = 0$ on $\chi \setminus \chi_{j+1}^{'}$.

Proof of Subclaim 1.

For each $j \in \mathbb{N}$, let us set $Y_j' = \{x \in \chi_{j+1}' \mid j + \frac{1}{2} \leq q(x) \leq j+1\}$. Then Y_j' and χ_j' are disjoint closed subsets of χ_{j+1}'. Applying Urysohn's lemma, we get that there exists $g_j : \chi_{j+1}' \to [0,1]$ continuous such that $g_j = 0$ on Y_j' and $g_j = 1$ on χ_j'. We can extend g_j to the whole χ by setting $g_j = 0$ on $\chi \setminus \chi_{j+1}'$. Then $f_j := f \cdot g_j$ is such that

- $0 \leq f_j \leq f$ on χ, since $0 \leq g_j \leq 1$ on χ.
- $f_j = f \cdot g_j = f$ on χ_j', since $g_j = 1$ on χ_j'.
- $f_j = f \cdot g_j = 0$ on $\chi \setminus \chi_{j+1}'$, since $g_j = 0$ on $\chi \setminus \chi_{j+1}'$.

In particular, $\text{supp}(f_j) \subseteq \chi_{j+1}'$ is compact, as closed subset of a compact set, and so $f_j \in \mathcal{C}_c(\chi)$.

$\square$(Subclaim 1)

Then $(f_j)_{j\in\mathbb{N}}$ is a non-decreasing sequence of non-negative functions in $C_c(\chi)$ which pointwise converges to f in χ. Indeed, for all $j \in \mathbb{N}$ and all $x \in \chi$, we easily get from Subclaim 1 that $0 \leq f_j(x) \leq f_{j+1}(x)$ and $\lim_{j\to\infty} f_j(x) = f(x)$. Hence, we can apply the Monotone Convergence Theorem, which ensures that

$$\int f d\mu = \lim_{j\to\infty} \int f_j d\mu \overset{(2.5)}{=} \lim_{j\to\infty} \overline{\overline{L}}(f_j).$$

Hence, the proof of the Main Claim is complete once we show that

Subclaim 2: $\overline{\overline{L}}(f) = \lim\limits_{j\to\infty} \overline{\overline{L}}(f_j)$.

Proof of Subclaim 2.

Let $j \in \mathbb{N}$. First of all, let us show that

$$\frac{q^2}{j} \geq f - f_j \geq 0 \text{ on } \chi. \tag{2.6}$$

From Subclaim 1 we know that $f = f_j$ on χ_j', so clearly $\dfrac{q^2}{j} \geq f - f_j = 0$ on χ_j'. Moreover, for any $x \in \chi \setminus \chi_j'$, we have $q(x) > j$ and so

$$q^2(x) > jq(x) = j\big(f(x) + \hat{p}(x)\big) \geq jf(x) \geq \big(f(x) - f_j(x)\big),$$

which yields $\dfrac{q^2(x)}{j} \geq (f - f_j)(x)$ for all $x \in \chi$.

Now (2.6) implies that $\overline{\overline{L}}\left(\frac{q^2}{j} - (f - f_j)\right) \geq 0$ and $\overline{\overline{L}}(f - f_j) \geq 0$. Hence, $\overline{\overline{L}}\left(\frac{q^2}{j}\right) \geq \overline{\overline{L}}(f - f_j) \geq 0$, i.e. $\frac{1}{j}\overline{\overline{L}}(q^2) \geq \overline{\overline{L}}(f - f_j) \geq 0$. Then passing to the limit for $j \to \infty$ we obtain that $\lim_{j\to\infty} \overline{\overline{L}}(f - f_j) = 0$ and so $\lim_{j\to\infty} \overline{\overline{L}}(f_j) = \overline{\overline{L}}(f)$.

□(Subclaim 2)

□(Main Claim)

Since $\hat{A} \subseteq \mathcal{B}(\chi)$, the Main Claim implies that for all $a \in A$ we have $\overline{\overline{L}}(\hat{a}) = \int \hat{a} d\mu$. This together with the definition of $\overline{L}$ and Claim 3 gives that

$$L(a) = \overline{L}(\hat{a}) = \overline{\overline{L}}(\hat{a}) = \int \hat{a} d\mu, \forall a \in A, \tag{2.7}$$

which yields the conclusion as μ is a finite Borel regular measure and so Radon. Indeed, using (2.7), we get that $L(1) = \int \hat{1} d\mu = \mu(\chi)$ and so that μ is finite.

□(Proof of Theorem 2.2.3)

2.3 Solving the KMP through characterizations of Psd(K)

The Riesz-Haviland theorem 2.2.1 establishes a beautiful duality between the $K-$moment problem and the problem of characterizing Psd(K). Hence, thanks to this result we can obtain necessary and sufficient conditions to solve the KMP using the characterizations of Psd(K) introduced in the previous chapter. For example, combining Riesz-Haviland's theorem with Theorem 1.3.9 about saturation of preorderings we obtain the following.

Corollary 2.3.1. *Let $L : \mathbb{R}[X] \to \mathbb{R}$ be linear and K a non-empty bcsas of $\mathbb{R}$ with natural description $S_{nat} = \{g_1, \dots, g_s\}$. Then there exists a $K-$representing measure for L if and only if $L(h^2 g_1^{e_1} \dots g_s^{e_s}) \geq 0$ for all $h \in \mathbb{R}[X]$ and all $e_1, \dots, e_s \in \{0, 1\}$.*

Proof.
By Theorem 2.2.1, the existence of a $K-$representing measure for L is equivalent to the non-negativity of L on Psd(K). The latter is in turn equivalent to the non-negativity of L on the preordering $T_{S_{nat}}$ associated to the natural description S_{nat} of K, since Theorem 1.3.9 ensures that Psd(K) $= T_{S_{nat}}$. Hence, the conclusion directly follows from the linearity of L as

$$T_{S_{nat}} = \left\{ \sum_{e=(e_1,\dots,e_s)\in\{0,1\}^s} \sigma_e\, g_1^{e_1} \dots g_s^{e_s} : \sigma_e \in \sum \mathbb{R}[X]^2, e \in \{0,1\}^s \right\}.$$

□

Corollary 2.3.1 allows to derive the most classical results about the one-dimensional KMP. Indeed, we have the following

- If $K = \mathbb{R}$, then $S_{nat} = \{\emptyset\}$ and so Corollary 2.3.1 becomes
  Theorem 2.3.2 (Hamburger [18])**.**
  A linear functional $L : \mathbb{R}[X] \to \mathbb{R}$ has a $\mathbb{R}-$representing measure if and only if $L(h^2) \geq 0$ for all $h \in \mathbb{R}[X]$.
- If $K = [0, +\infty)$, then $S_{nat} = \{X\}$ and so Corollary 2.3.1 becomes
  Theorem 2.3.3 (Stieltjes [52])**.**
  A linear functional $L : \mathbb{R}[X] \to \mathbb{R}$ has a $\mathbb{R}^+-$representing measure if and only if $L(h^2) \geq 0$ and $L(Xh^2) \geq 0$ for all $h \in \mathbb{R}[X]$.
- If $K = [0, 1]$, then $S_{nat} = \{X, 1 - X\}$. Hence, using Corollary 2.3.1 together with the observation that $X(1-X) = X(1-X)^2 + (1-X)X^2$, we obtain
  Theorem 2.3.4 (Hausdorff [19])**.**
  A linear functional $L : \mathbb{R}[X] \to \mathbb{R}$ has a $[0,1]-$representing measure if and only if $L(h^2) \geq 0$, $L(Xh^2) \geq 0$ and $L((1-X)h^2) \geq 0$ for all $h \in \mathbb{R}[X]$.

These classical results were obtained without using Riesz-Haviland theorem, but through methods involving the analysis of the so-called *Hankel matrix* or *moment matrix* associated to the starting functional. In fact, we will see that any condition of the form $L(gh^2) \geq 0$ for all $h \in \mathbb{R}[X]$ and some $g \in \mathbb{R}[X]$ can be translated into the positive semidefiniteness of a certain matrix obtained from the putative moment sequence $(L(X^j))_{j \in \mathbb{N}_0}$.

Let us introduce these concepts for any dimension $n \in \mathbb{N}$.

Definition 2.3.5. *A sequence $m := (m_\alpha)_{\alpha \in \mathbb{N}_0^n}$ of real numbers is called* positive semidefinite (psd) *if*

$$\sum_{\alpha, \beta \in F} c_\alpha c_\beta m_{\alpha+\beta} \geq 0, \quad \forall\ F \subset \mathbb{N}_0^n,\ c_\alpha,\ c_\beta \in \mathbb{R}.$$

Definition 2.3.6. *Given a polynomial $g := \sum_{\gamma \in \mathbb{N}_0^n} g_\gamma \underline{X}^\gamma \in \mathbb{R}[X_1, \ldots, X_n]$ and a sequence $m := (m_\alpha)_{\alpha \in \mathbb{N}_0^n}$ of real numbers, we define $g(E)m := \big((g(E)m)_\alpha\big)_{\alpha \in \mathbb{N}_0^n}$, where*

$$(g(E)m)_\alpha := \sum_{\gamma \in \mathbb{N}_0^n} g_\gamma m_{\alpha+\gamma}.$$

Examples 2.3.7.

1. *For $m := (m_j)_{j \in \mathbb{N}_0} = (m_0, m_1, m_2, \ldots)$, $g := X$ and $h := X^3 - 1$ we get:*
   $g(E)m = (m_{j+1})_{j \in \mathbb{N}_0} = (m_1, m_2, m_3, \ldots)$ and
   $h(E)m = (m_{j+3} - 1)_{j \in \mathbb{N}_0} = (m_3 - 1, m_4 - 1, m_5 - 1, \ldots)$.

2. *For* $m := \left(m_{(\alpha_1,\alpha_2)}\right)_{(\alpha_1,\alpha_2)\in\mathbb{N}_0^2}$ *and* $g := 5 - X_1^2 - X_2^2$, *we have that* $(g(E)m)_{(\alpha_1,\alpha_2)} = 5m_{(\alpha_1,\alpha_2)} - m_{(\alpha_1+2,\alpha_2)} - m_{(\alpha_1,\alpha_2+2)}$.
   For instance, $(g(E)m)_{(0,1)} = 5m_{(0,1)} - m_{(2,1)} - m_{(0,3)}$.

Lemma 2.3.8.
Given $L : \mathbb{R}[X_1,\ldots,X_n] \to \mathbb{R}$ *linear and* $g := \sum_{\gamma\in\mathbb{N}_0^n} g_\gamma \underline{X}^\gamma \in \mathbb{R}[X_1,\ldots,X_n]$, *we have that* $L(gh^2) \geq 0, \forall h \in \mathbb{R}[X_1,\ldots,X_n]$ *if and only if* $g(E)m$ *is psd, where* $m := (L(\underline{X}^\alpha))_{\alpha\in\mathbb{N}_0^n}$.

Proof.
For any $\alpha \in \mathbb{N}_0^n$, we have

$$L(g\underline{X}^\alpha) = L\left(\sum_{\gamma\in\mathbb{N}_0^n} g_\gamma \underline{X}^{\gamma+\alpha}\right) = \sum_{\gamma\in\mathbb{N}_0^n} g_\gamma L(\underline{X}^{\gamma+\alpha}) = \sum_{\gamma\in\mathbb{N}_0^n} g_\gamma m_{\gamma+\alpha} = (g(E)m)_\alpha\,.$$

Let $h = \sum_{\beta\in\mathbb{N}_0^n} h_\beta \underline{X}^\beta \in \mathbb{R}[\underline{X}]$. Then $h^2 = \sum_{\beta,\gamma\in\mathbb{N}_0^n} h_\beta h_\gamma \underline{X}^{\beta+\gamma}$ and so

$$\begin{aligned} L(gh^2) &= L\left(g \sum_{\beta,\gamma\in\mathbb{N}_0^n} h_\beta h_\gamma \underline{X}^{\beta+\gamma}\right) \\ &= \sum_{\beta,\gamma\in\mathbb{N}_0^n} h_\beta h_\gamma L(g\underline{X}^{\beta+\gamma}) \\ &= \sum_{\beta,\gamma\in\mathbb{N}_0^n} h_\beta h_\gamma \left(g(E)m\right)_{\beta+\gamma}. \end{aligned}$$

Hence, $L(gh^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$ iff $\sum_{\beta,\gamma\in\mathbb{N}_0^n} h_\beta h_\gamma \left(g(E)m\right)_{\beta+\gamma} \geq 0$ for all $h_\beta, h_\gamma \in \mathbb{R}$, which is equivalent the psd-ness of $g(E)m$. □

Definition 2.3.9. *Let* $L : \mathbb{R}[\underline{X}] \longrightarrow \mathbb{R}$ *be linear and* $g \in \mathbb{R}[\underline{X}]$. *We define the associated symmetric bilinear form as*

$$\begin{aligned} \langle\ ,\ \rangle_g :\ \mathbb{R}[\underline{X}] \times \mathbb{R}[\underline{X}] &\to \mathbb{R} \\ (\ p\ ,\ q\ ) &\mapsto \langle p,q\rangle_g := L(pqg) \end{aligned}$$

The moment matrix *associated to* L *and* localized *at* g *is defined to be the infinite real symmetric matrix* $M^g := \left(\langle\underline{X}^\alpha, \underline{X}^\beta\rangle_g\right)_{\alpha,\beta\in\mathbb{N}_0^n} = \left(L(\underline{X}^{\alpha+\beta}\ g)\right)_{\alpha,\beta\in\mathbb{N}_0^n}$. *For* $g = 1$, M^1 *is just said the moment matrix associated to* L.

Examples 2.3.10.

a) *Let* $n = 1$*,* $L : \mathbb{R}[X] \to \mathbb{R}$ *linear and set* $m := (m_j)_{j \in \mathbb{N}_0}$ *with* $m_j := L(X^j)$*. Then the associated moment matrix is*

$$M^1 = \begin{pmatrix} m_0 & m_1 & m_2 & \dots \\ m_1 & m_2 & \iddots & \dots \\ m_2 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} L(1) & L(X) & L(X^2) & \dots \\ L(X) & L(X^2) & \ddots & \dots \\ L(X^2) & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

If $g := X$ *then the corresponding localized moment matrix is given by*

$$M^g = \begin{pmatrix} m_1 & m_2 & m_3 & \dots \\ m_2 & m_3 & \iddots & \ddots \\ m_3 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} L(X) & L(X^2) & L(X^3) & \dots \\ L(X^2) & L(X^3) & \ddots & \ddots \\ L(X^3) & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

b) *Let* $n = 2$*,* $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ *linear and set* $m := (m_\alpha)_{\alpha \in \mathbb{N}_0^2}$ *with* $m_{(\alpha_1, \alpha_2)} := L(X_1^{\alpha_1} X_2^{\alpha_2})$*. Then the associated moment matrix is*

$$\begin{aligned} M^1 &= \begin{pmatrix} m_{00} & m_{10} & m_{01} & m_{20} & m_{11} & \dots \\ m_{10} & m_{20} & m_{11} & m_{30} & \ddots & \ddots \\ m_{01} & m_{11} & m_{20} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \\ &= \begin{pmatrix} L(1) & L(X_1) & L(X_2) & L(X_1^2) & L(X_1X_2) & \dots \\ L(X_1) & L(X_1^2) & L(X_1X_2) & L(X_1^3) & \ddots & \ddots \\ L(X_2) & L(X_1X_2) & L(X_1^3) & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \end{aligned}$$

and if $g = X_1X_2$ *then the corresponding localized moment matrix is*

$$M^g = \begin{pmatrix} m_{11} & m_{21} & m_{12} & m_{31} & m_{22} & \dots \\ m_{21} & m_{31} & m_{22} & m_{41} & \ddots & \ddots \\ m_{12} & m_{22} & m_{31} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Recall that

Definition 2.3.11. *A real symmetric $N \times N$ matrix A is* positive semidefinite (psd) *if* $\underline{y}^t A \underline{y} \geq 0 \ \forall \ \underline{y} \in \mathbb{R}^N$. *An infinite real symmetric matrix A is* psd *if* $\underline{y}^t A_N \underline{y} \geq 0 \ \forall \ \underline{y} \in \mathbb{R}^N$ *and* $\forall \ N \in \mathbb{N}$, *where A_N is the upper left corner submatrix of order N of A.*

Proposition 2.3.12. *Let $L : \mathbb{R}[\underline{X}] \longrightarrow \mathbb{R}$ be linear and $g \in \mathbb{R}[\underline{X}]$. Then the following are equivalent:*
1) $L(\sigma g) \geq 0 \ \forall \ \sigma \in \sum \mathbb{R}[\underline{X}]^2$.
2) $L(h^2 g) \geq 0 \ \forall \ h \in \mathbb{R}[\underline{X}]$.
3) $\langle \ , \ \rangle_g$ *is psd.*
4) M^g *is psd.*
5) $g(E)m$ *is psd where* $m := (L(\underline{X}^\alpha))_{\alpha \in \mathbb{N}_0^n}$.

Proof.
1) $\Leftrightarrow$ 2) since for any $\sigma \in \sum \mathbb{R}[\underline{X}]^2$, there exist $h_i \in \mathbb{R}[\underline{X}]$ such that $\sigma = \sum_i h_i^2$ and so $L(\sigma g) = \sum_i L(h_i^2 g)$.

2) $\Leftrightarrow$ 3) as $L(h^2 g) = \langle h, h \rangle_g$

3) $\Leftrightarrow$ 4) Indeed, for any $h = \sum_{\gamma \in F} h_\gamma \underline{X}^\gamma \in \mathbb{R}[\underline{X}]$ with $F \subset \mathbb{N}_0^n$ finite, we have

$$\begin{aligned} \langle h, h \rangle_g &= L(\sum_{\beta,\gamma \in F} h_\beta h_\gamma \underline{X}^{\beta+\gamma} g) = \sum_{\beta,\gamma \in F} h_\beta h_\gamma L(g \underline{X}^{\beta+\gamma}) \\ &= \sum_{\beta,\gamma \in F} h_\beta h_\gamma M^g(\beta, \gamma) = y^t M^g_{|F|} y, \end{aligned}$$

where $y := (h_\gamma)_{\gamma \in F}$.

4) $\Leftrightarrow$ 5)
$g(E)m$ is psd iff $\sum_{\beta,\gamma \in \mathbb{N}_0^n} h_\beta h_\gamma \left(g(E)m\right)_{\beta+\gamma} \geq 0$ for all $h_\beta, h_\gamma \in \mathbb{R}$, which is equivalent to the psd-ness of M^g since $\left(g(E)m\right)_{\beta+\gamma} = M^g(\beta, \gamma)$.

5) $\Leftrightarrow$ 1) by Lemma 2.3.8. □

We can then express the Hambuger, Stieltjes and Hausdorff solutions to the KMP in terms of moment matrices.

Theorem 2.3.13.
Given $m := (m_j)_{j\in\mathbb{N}_0}$, the following are equivalent:
a) m is a Hamburger's moment sequence, i.e. has a $\mathbb{R}-$representing measure
b) m is psd
c) M^1 is psd
d) $L_m(h^2) \geq 0$ for all $h \in \mathbb{R}[X]$.

Theorem 2.3.14.
Given $m := (m_j)_{j\in\mathbb{N}_0}$, the following are equivalent:
a) m is a Stieltjes's moment sequence, i.e. has a $\mathbb{R}^+-$representing measure
b) m and $g(E)m$ are both psd
c) M^1 and M^g are both psd
d) $L_m(h^2) \geq 0$ and $L_m(gh^2) \geq 0$ for all $h \in \mathbb{R}[X]$.
where $g := X$.

Theorem 2.3.15.
Given $m := (m_j)_{j\in\mathbb{N}_0}$, the following are equivalent:
a) m is a Hausdorff's moment sequence, i.e. has a $[0,1]-$representing measure
b) m, $g_1(E)m$ and $g_2(E)m$ are all psd
c) M^1, M^{g_1} and M^{g_2} are all psd
d) $L(h^2) \geq 0$, $L(g_1h^2) \geq 0$ and $L(g_2h^2) \geq 0$ for all $h \in \mathbb{R}[X]$.
where $g_1 := X$ and $g_2 := 1 - X$.

Let us now relate to the KMP the Nichtnegativstellensätze and the closure results introduced in the previous chapter.

Proposition 2.3.16.
Let τ be a locally convex topology on $\mathbb{R}[\underline{X}]$. Given a convex cone C of $\mathbb{R}[\underline{X}]$ and a closed subset K of $\mathbb{R}^n$, the following are equivalent
a) $\mathrm{Psd}(K) \subseteq C_\tau^{\vee\vee}$
b) $\forall\ L \in C_\tau^{\vee}$, $\exists\ \mu$ $K-$representing measure for L,
where:
$C_\tau^{\vee} := \{\ell : \mathbb{R}[\underline{X}] \to \mathbb{R}$ *linear* $|\ell$ *is* $\tau -$ *continuous and* $\ell(C) \geq 0\}$ *and*
$C_\tau^{\vee\vee} := \{p \in \mathbb{R}[\underline{X}]\,|\forall\,\ell \in C_\tau^{\vee},\, \ell(p) \geq 0\}$.

Proof.
a) $\Rightarrow$ *b)* Let $L \in C_\tau^{\vee}$, i.e. L is $\tau-$ continuous and non-negative on C. Then $L\left(\overline{C}^{\tau}\right) \subseteq [0,+\infty)$ and so, by Corollary 1.3.35, $L\left(C_\tau^{\vee\vee}\right) \subseteq [0,+\infty)$. This implies by a) that $L(\mathrm{Psd}(K)) \subseteq [0,+\infty)$ which is equivalenty by Riesz-Haviland Theorem 2.2.1 to the existence of a $K-$representing measure for L.

b) $\Rightarrow$ *a)* By b), we have that $\forall\ L \in C_\tau^{\vee}$, $L(\mathrm{Psd}(K)) \subseteq [0,+\infty)$, i.e. $L \in (\mathrm{Psd}(K))_\tau^{\vee}$. Then $C_\tau^{\vee} \subseteq (\mathrm{Psd}(K))_\tau^{\vee}$ and so

$$C_\tau^{\vee\vee} \supseteq (\mathrm{Psd}(K))_\tau^{\vee\vee} \supseteq \mathrm{Psd}(K).$$

□

By combining the previous result with Corollary 1.3.42 (respectively 1.3.41 and 1.3.40) and recalling that every linear functional is continuous w.r.t. the finest locally convex topology, we obtain the following results for the KMP.

Corollary 2.3.17. *Let $L:\mathbb{R}[\underline{X}]\to\mathbb{R}$ linear and $S:=\{g_1,\ldots,g_s\}\subset\mathbb{R}[\underline{X}]$ such that the associated bcsas K_S is compact. Then there exists a K_S-representing measure for L if and only if $L(h^2g_1^{e_1}\cdots g_s^{e_s})\geq 0$ for all $h\in\mathbb{R}[\underline{X}]$, $e_1,\ldots,e_s\in\{0,1\}$.*

Corollary 2.3.18. *Let $L:\mathbb{R}[\underline{X}]\to\mathbb{R}$ linear and $S:=\{g_1,\ldots,g_s\}\subset\mathbb{R}[\underline{X}]$ such that the quadratic module M_S generated by S is Archimedean. Then there exists a K_S-representing measure for L if and only if $L(h^2g_i)\geq 0$ for all $h\in\mathbb{R}[\underline{X}]$ and $i\in\{0,1,\ldots,s\}$, where $g_0:=1$.*

Corollary 2.3.19. *Let $L:\mathbb{R}[\underline{X}]\to\mathbb{R}$ linear and M be an Archimedean $2d-$power module of $\mathbb{R}[\underline{X}]$ with $d\in\mathbb{N}$. Then $\exists$ a K_M-representing measure for L if and only if $L(M)\subseteq[0,+\infty)$.*

Remark 2.3.20. *Corollary 2.3.17 is actually the dual facet of Corollary 1.3.42, since we can also deduce Corollary 1.3.42 from Corollary 2.3.17. Indeed, by Proposition 2.3.16, Corollary 2.3.17 is equivalent to $\mathrm{Psd}(K_S)\subseteq(T_S)^{\vee\vee}_\tau$. This together with Corollary 1.3.35 and the fact that $\mathrm{Psd}(K_S)=\bigcap_{x\in K_S}e_x^{-1}([0,+\infty))$ (where $e_x(p):=p(x)$ for all $p\in\mathbb{R}[\underline{X}]$) yields that*

$$\mathrm{Psd}(K_S)\subseteq(T_S)^{\vee\vee}_\varphi=\overline{T_S}^{\varphi}\subseteq\overline{\mathrm{Psd}(K_S)}^{\varphi}=\mathrm{Psd}(K_S).$$

Hence, $\mathrm{Psd}(K_S)=\overline{T_S}^{\varphi}$, i.e. Corollary 1.3.42 holds.

A similar argument shows that Corollary 1.3.41 (respectively, Corollary 1.3.40) can be derived from Corollary 2.3.18 (respectively, Corollary 2.3.19).

Proposition 2.3.16 can be easily generalized to any unital commutative $\mathbb{R}-$algebra with the only additional assumption that

$$\exists\, p\in A, \text{s.t. } \hat{p}\geq 0 \text{ on } K \text{ and } \forall\, n\in\mathbb{N},\ \{\alpha\in K:\hat{p}(\alpha)\leq n\} \text{ is compact.} \tag{2.8}$$

This hypothesis is fundamental for the application of the generalized Riesz-Haviland Theorem 2.2.2 and so to get the following.

Proposition 2.3.21. *Let A be a unital commutative $\mathbb{R}-$algebra and C a convex cone of A. Given a locally convex topology τ on A and a closed subset K of $X(A)$ s.t. (2.8) holds, the following are equivalent*
a) $\mathrm{Psd}(K)\subseteq C^{\vee\vee}_\tau$
b) $\forall\ L\in C^\vee_\tau$, $\exists\ \mu$ *$K-$representing measure for L,*
where:
$C^\vee_\tau:=\{\ell:A\to\mathbb{R}\ \textit{linear}|\ell\ \textit{is}\ \tau-\textit{continuous and}\ \ell(C)\geq 0\}$ *and*
$C^{\vee\vee}_\tau:=\{a\in A\,|\forall\,\ell\in C^\vee_\tau,\,\ell(a)\geq 0\}$.

Proof.

a) $\Rightarrow$ *b)* Let $L \in C_\tau^\vee$, i.e. L is $\tau-$continuous and non-negative on C. Then $L\left(\overline{C}^\tau\right) \subseteq [0,+\infty)$ and so, by Corollary 1.3.35, $L\left(C_\tau^{\vee\vee}\right) \subseteq [0,+\infty)$. This implies by a) that $L(\mathrm{Psd}(K)) \subseteq [0,+\infty)$ which is equivalent by generalized Riesz-Haviland Theorem 2.2.2 to the existence of a $K-$representing measure for L. Note that we can apply the generalized Riesz-Haviland Theorem 2.2.2 since we assumed that (2.8) holds.

b) $\Rightarrow$ *a)* By b), we have that for any $L \in C_\tau^\vee$ there exists a non-negative Radon measure μ supported in K and such that $L(a) = \int \hat{a} d\mu$. Hence, for all $a \in \mathrm{Psd}(K)$ we have $L(a) \geq 0$, i.e. $L \in (\mathrm{Psd}(K))_\tau^\vee$. Then $C_\tau^\vee \subseteq (\mathrm{Psd}(K))_\tau^\vee$ and so $C_\tau^{\vee\vee} \supseteq (\mathrm{Psd}(K))_\tau^{\vee\vee} \supseteq \mathrm{Psd}(K)$. □

By combining Proposition 2.3.21 with Theorem 1.3.45 we get the following result for Problem 2.1.6.

Theorem 2.3.22. *Let (A,ρ) be a unital commutative seminormed $\mathbb{R}-$algebra, $L: A \to \mathbb{R}$ linear, $d \in \mathbb{N}$ and M a $2d-$power module of A. Then there exists a $(K_M \cap \mathfrak{sp}(\rho))-$representing measure for L if and only if L is $\rho-$continuous and $L(M) \subseteq [0,\infty)$.*

Before proving it, let us recall that the Gelfand spectrum $\mathfrak{sp}(\rho)$ is the collection of all $\rho-$continuous characters of A and let us show the following property.

Lemma 2.3.23. *If (A,ρ) is a unital commutative seminormed $\mathbb{R}-$algebra, then the Gelfand spectrum $\mathfrak{sp}(\rho)$ is compact.*

Proof. By Lemma 2.3.8, we know that

$$\begin{aligned} \mathfrak{sp}(\rho) &= \{\alpha \in X(A) : |\alpha(a)| \leq \rho(a),\ \forall\ a \in A\} \\ &= \left\{\alpha \in X(A) : (\hat{a}(\alpha))_{a\in A} \in \prod_{a\in A}[-\rho(a),\rho(a)]\right\}. \end{aligned}$$

Hence, using the embedding

$$\begin{array}{rccl} \pi: & X(A) & \to & \mathbb{R}^A \\ & \alpha & \mapsto & \pi(\alpha) := (\alpha(a))_{a\in A} = (\hat{a}(\alpha))_{a\in A}. \end{array}$$

we have that

$$\pi(\mathfrak{sp}(\rho)) = \pi(X(A)) \cap \prod_{a\in A}[-\rho(a),\rho(a)] \tag{2.9}$$

Since $\pi(X(A))$ is closed in $(\mathbb{R}^A, \tau_{prod})$ (see Sheet 5) and $\prod_{a\in A}([-\rho(a), \rho(a)])$ is compact in $(\mathbb{R}^A, \tau_{prod})$ by Tychonoff theorem, (2.9) ensures that $\pi(\mathfrak{sp}(\rho))$ is a closed subset of a compact set and so it is compact itself.

Let $(U_i)_{i\in I}$ s.t. $U_i \in \tau_{X(A)}$ and $\mathfrak{sp}(\rho) \subseteq \bigcup_{i\in I} U_i$. Then by Remark 2.1.5 for each $i \in I$ there exists $O_i \in \tau_{prod}$ s.t. $\pi^{-1}(O_i) = U_i$. Hence,

$$\mathfrak{sp}(\rho) \subseteq \bigcup_{i\in I} \pi^{-1}(O_i) = \pi^{-1}\left(\bigcup_{i\in I} O_i\right),$$

which implies $\pi(\mathfrak{sp}(\rho)) \subseteq \pi\left(\pi^{-1}\left(\bigcup_{i\in I} O_i\right)\right) \subseteq \bigcup_{i\in I} O_i$. Then the compactness of $\pi(\mathfrak{sp}(\rho))$ guarantees that there exists $J \subset I$ finite and such that $\pi(\mathfrak{sp}(\rho)) \subseteq \bigcup_{i\in J} O_i$, which gives

$$\mathfrak{sp}(\rho) \subseteq \pi^{-1}\left(\pi(\mathfrak{sp}(\rho))\right) \subseteq \pi^{-1}\left(\bigcup_{i\in J} O_i\right) = \bigcup_{i\in J} \pi^{-1}(O_i) = \bigcup_{i\in J} U_i.$$

Hence, $\mathfrak{sp}(\rho)$ is compact. □

Proof. of Theorem 2.3.22
Since (A, ρ) is a seminormed algebra (and so in particular a locally convex t.v.s.) we can apply both Theorem 1.3.45 and Corollary 1.3.35, which yield

$$\mathrm{Psd}(K_M \cap \mathfrak{sp}(\rho)) = \overline{M}^{\rho} = M_{\rho}^{\vee\vee}.$$

Moreover, (2.8) holds by taking $p = 1$. Indeed, $\hat{1} = 1 > 0$ on $X(A)$ and for all $n \in \mathbb{N}$ the set $\{\alpha \in K_M \cap \mathfrak{sp}(\rho) : \hat{1}(\alpha) \leq n\}$ is nothing but $K_M \cap \mathfrak{sp}(\rho)$ which is compact by Lemma 2.3.23.

Suppose that L is $\rho-$continuous and $L(M) \subseteq [0, \infty)$, i.e. $L \in M_{\rho}^{\vee}$. Then Proposition 2.3.21 ensures that there exists a $(K_M \cap \mathfrak{sp}(\rho))-$representing measure for L.

Conversely, suppose that there exists a $(K_M \cap \mathfrak{sp}(\rho))-$representing measure for L. Then clearly $L(M) \subseteq [0, +\infty)$ and for any $a \in A$ we have that

$$|L(a)| \leq \int_{K_M \cap \mathfrak{sp}(\rho)} |\hat{a}(\alpha)|\, d\mu(\alpha) \leq \rho(a) L(1),$$

i.e. L is $\rho-$continuous. □

Remark 2.3.24. *Theorem 2.3.22 is actually the dual facet of Theorem 1.3.45, since we can also deduce Theorem 1.3.45 from Theorem 2.3.22. Indeed, we have already observed that (2.8) holds because of the compactness of* $K_M \cap \mathfrak{sp}(\rho)$

and so we can apply Proposition 2.3.21, which ensures that Theorem 2.3.22 is equivalent to $\mathrm{Psd}(K_M \cap \mathfrak{sp}(\rho)) \subseteq M_\rho^{\vee\vee}$*. This together with Corollary 1.3.35 and the fact that* $\mathrm{Psd}(K_M \cap \mathfrak{sp}(\rho)) = \bigcap_{\alpha \in K_M \cap \mathfrak{sp}(\rho)} \alpha^{-1}([0, +\infty))$ *yields that*

$$\mathrm{Psd}(K_M \cap \mathfrak{sp}(\rho)) \subseteq M_\rho^{\vee\vee} = \overline{M}^\rho \subseteq \overline{\mathrm{Psd}(K_M \cap \mathfrak{sp}(\rho))}^\rho = \mathrm{Psd}(K_M \cap \mathfrak{sp}(\rho)).$$

Hence, $\mathrm{Psd}(K_M \cap \mathfrak{sp}(\rho)) = \overline{M}^\rho$*, i.e. Theorem 1.3.45 holds.*

Theorem 2.3.22 easily extends to the case when A is an arbitrary lmc algebra (i.e. a topological algebra, where the the topology is generated by a family of submultiplicative seminorms).

Theorem 2.3.25. *Let* (A, τ) *be a unital commutative lmc* $\mathbb{R}-$*algebra,* $d \in \mathbb{N}$*,* M *a* $2d-$*power module of* A *and* $L : A \to \mathbb{R}$ *linear. Then* L *is* $\tau-$*continuous and* $L(M) \subseteq [0, \infty)$ *if and only if there exists a* $(K_M \cap \mathfrak{sp}(\rho))-$*representing measure for* L *for some* $\rho \in \mathcal{F}$*, where* $\mathcal{F}$ *is a directed family of submultiplicative seminorms generating* τ*.*

Proof. Since (A, τ) is an lmc algebra, there always exists a directed family $\mathcal{F}$ of submultiplicative seminorms generating τ (see [21, Theorem 4.2.14]). Then the $\tau-$continuity of L is equivalent to the $\rho-$continuity of L for some $\rho \in \mathcal{F}$ by [21, Proposition 4.6.1]. Hence, Theorem 2.3.22 guarantees that there exists a $(K_M \cap \mathfrak{sp}(\rho))-$representing measure for L. □

In Theorems 2.3.22 and 2.3.25 as well as in Corollaries 2.3.17, 2.3.18 and 2.3.19 the representing measure are always compactly supported. This gives in turn the uniqueness of the representing measure in each of these results.

Theorem 2.3.26. *If* μ *is a Radon measure on* $X(A)$ *supported on a compact subset* K*, then it is determinate, i.e. any other Radon measure* ν *on* $X(A)$ *such that* $\int \hat{a} d\mu = \int \hat{a} d\nu$ *for all* $a \in A$ *coincides with* μ*.*

To prove this result we will make use of the Stone-Weirstrass Theorem, which we state here for the convenience of the reader.

Theorem 2.3.27 (Stone-Weirstrass' Theorem)**.** *Let* χ *be a Hausdorff compact topological space and* $\mathcal{C}$ *a subalgebra of* $\mathcal{C}(\chi)$ *containing a non-zero constant function. Then* $\mathcal{C}$ *is dense in* $\mathcal{C}(\chi)$ *if and only if* $\mathcal{C}$ *separates the points of* χ*, i.e. for any* $x \neq y$ *in* χ *there exists* $f \in \mathcal{C}$ *such that* $f(x) \neq f(y)$*.*

Proof. of Theorem 2.3.26
Let us first show that ν is also supported in K and then that ν coincides with μ.

Suppose that ν is not supported in K. Then there exists $Z \subseteq X(A) \setminus K$ compact and such that $\nu(Z) > 0$. Let $\varepsilon > 0$ such that $\varepsilon < \frac{\nu(Z)}{\mu(K)+\nu(Z)}$. Now $\{\hat{a} : a \in A\}$ is a subalgebra of $\mathcal{C}(X(A))$ which separates the points of $X(A)$, since for any $\alpha_1 \neq \alpha_2$ in $X(A)$ there exists $a \in A$ such that $\alpha_1(a) \neq \alpha_2(a)$, i.e. $\hat{a}(\alpha_1) \neq \hat{a}(\alpha_2)$. Hence, $\{\hat{a} : a \in A\}$ in particular separates the points of $K \cup Z$. Since K and Z are both compact and disjoint, we can apply Urysohn's lemma, which ensures that there exists $g \in \mathcal{C}(K \cup Z)$ such that $g \restriction_K = 0$ and $g \restriction_Z = 1$. Therefore, by Stone-Weirstrass' Theorem 2.3.27 applied to $K \cup Z$, we obtain that there exists $a \in A$ such that $|\hat{a}(\alpha) - g(\alpha)| \leq \varepsilon, \ \forall \ \alpha \in K \cup Z$, i.e.

$$\exists \ a \in A : |\hat{a}(\alpha)| \leq \varepsilon, \ \forall \ \alpha \in K \text{ and } |\hat{a}(\alpha) - 1| \leq \varepsilon, \ \forall \ \alpha \in Z.$$

W.l.o.g. we can assume $\hat{a} \geq 0$ on $X(A)$ (otherwise replace a with a^2). Then we have

$$(1-\varepsilon)\nu(Z) \leq \int |\hat{a}| d\nu \leq \int \hat{a} d\nu = \int \hat{a} d\mu \leq \int |\hat{a}| d\mu \leq \varepsilon\mu(K),$$

which yields $\nu(Z) \leq \varepsilon\left(\mu(Z) + \nu(Z)\right) < \nu(Z)$ and so a contradiction. Hence, ν is also supported in K and so we have that $\int_K \hat{b} d\mu = \int_K \hat{b} d\nu, \ \forall \ b \in A$. Hence, by Stone-Weierstrass' Theorem 2.3.27, we get $\int_K \varphi d\mu = \int_K \varphi d\nu, \ \forall \ \varphi \in \mathcal{C}(K)$. Then $\mu = \nu$ by the uniqueness in Riesz-Markov-Kakutani Representation Theorem 2.2.5. $\square$

Chapter 3

$K-$Moment Problem: the operator theoretical approach

3.1 Basics from spectral theory

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space (i.e. a complete inner product space). We denote by $\|\cdot\|$ the norm induced on $\mathcal{H}$ by the inner product $\langle\cdot,\cdot\rangle$.

Definition 3.1.1. *An* operator *T on $\mathcal{H}$ is a linear map from a linear subspace $\mathcal{D}(T)$ of $\mathcal{H}$ (called the* domain *of T) into $\mathcal{H}$. We say that*

- *T is* bounded *if its operator norm $\|T\|_{op} := \sup_{x\in\mathcal{D}(T)\setminus\{o\}} \frac{\|Tx\|}{\|x\|}$ is finite.*
- *T is* symmetric *if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in \mathcal{D}(T)$.*

3.1.1 Bounded operators

In this subsection we are going to focus on bounded operators defined everywhere in $\mathcal{H}$.

Definition 3.1.2. *Let T be a bounded operator with $\mathcal{D}(T) = \mathcal{H}$. Then*

- *the unique bounded operator $T^* : \mathcal{H} \to \mathcal{H}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in \mathcal{H}$ is called the* adjoint *of T.*
- *T is called* self-adjoint *if $T = T^*$.*

Note that a bounded operator defined everywhere in $\mathcal{H}$ is self-adjoint if and only if it is symmetric.

Definition 3.1.3. *Two operators T_1, T_2 defined on the same Hilbert space $\mathcal{H}$* commute *if $T_1T_2x = T_2T_1x$ for all $x \in \mathcal{H}$.*

Theorem 3.1.4 (Spectral Theorem for bounded operators)**.** *Let $T_1, \ldots, T_n$ be n pairwise commuting bounded self-adjoint operators having as domain the* *same separable Hilbert space $\mathcal{H}$ and let $v \in \mathcal{H}$. Then there exists a unique non-negative Radon measure μ_v on $\mathbb{R}^n$ such that*

$$\langle v, T_1^{\alpha_1} \cdots T_n^{\alpha_n} v\rangle = \int_{\mathbb{R}^n} \underline{X}^\alpha d\mu_v < \infty, \ \forall \ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$$

and μ_v is supported in $B_{\|T_1\|_{op}}(0) \times \cdots \times B_{\|T_n\|_{op}}(0)$ where $B_R(0)$ denotes the closed ball of radius R and center 0 in $\mathbb{R}$.

(for a proof see e.g. [44, Chapter VII] and [49, Theorem 5.23]).

Let us also recall a fundamental theorem about linear transformations on normed spaces (see e.g. [44, Theorem I.7]), which will be useful in the following.

Theorem 3.1.5 (Bounded Linear Transformation Theorem)**.** *Let Y be a Banach space, Z be a normed space, and U a dense subset of Z. If $\varphi : U \to Y$ is a bounded linear map, then φ can be uniquely extended to a bounded linear map $\overline{\varphi} : Z \to Y$ and $\|\overline{\varphi}\|_{op} = \|\varphi\|_{op}$*

3.1.2 Unbounded operators

By the Hellinger-Toeplitz theorem, a symmetric operator T with $\mathcal{D}(T) = \mathcal{H}$ is always bounded (see e.g. [44, Section III.5]). Hence, unbounded symmetric operators cannot be defined everywhere in $\mathcal{H}$. For this reason, we need a more general definition of adjoint than the one given for bounded operators.

Definition 3.1.6. *Let $T : \mathcal{D}(T) \to \mathcal{H}$ be linear with $\mathcal{D}(T)$ dense[1] in $\mathcal{H}$. Then*

- *the* adjoint *of T is the linear operator T^* with domain*

$$\mathcal{D}(T^*) := \{w \in \mathcal{H} : \exists z_w \in \mathcal{H} \ s.t. \ \langle Tv, w\rangle = \langle v, z_w\rangle, \quad \forall v \in \mathcal{D}(T)\}$$

  defined by $T^* v = z_w$ for all $v \in \mathcal{D}(T^*)$.
- *T is called* self-adjoint *if $T = T^*$.*

Definition 3.1.7.
Let T_1 and T_2 be two self-adjoint operators with domain in the same Hilbert space $\mathcal{H}$. We say that T_1 and T_2 are strongly commuting *if $e^{ir_1 T_1} e^{ir_2 T_2} = e^{ir_2 T_2} e^{ir_1 T_1}$ for all $r_1, r_2 \in \mathbb{R}$.*

Theorem 3.1.8 (Spectral Theorem for unbounded operators)**.**
Let $(T_1, \ldots, T_n)$ be a tuple of self-adjoint operators with domain dense in the same separable Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ which are pairwise strongly commuting

[1] The density of $\mathcal{D}(T)$ in $\mathcal{H}$ ensures that z_w is uniquely determined by the equation $\langle Tv, w\rangle = \langle v, z_w\rangle, \quad \forall v \in \mathcal{D}(T)$.

and let $v \in \mathcal{H}$ be such that $\forall d \in \mathbb{N}_0$, $\forall\, i_1, \ldots, i_{d+1} \in \{1, \ldots, n\}$ we have $T_{i_d} \cdot T_{i_{d-1}} \cdots T_{i_1} v \in \mathcal{D}(T_{i_{d+1}})$ (for $d = 0$ we set T_{i_0} to be the identity operator). Then there exists a unique non-negative Radon measure μ_v such that

$$\langle v, T_{i_d} \cdot T_{i_{d-1}} \cdots T_{i_1} v\rangle = \int_{\mathbb{R}^n} X_{i_1} \cdots X_{i_d} d\mu_v, \ \forall\, d \in \mathbb{N}_0, i_1, \ldots, i_d \in \{1, \ldots, n\}$$

(for a proof see e.g. [44, Section VIII.3] and [49, Theorem 5.23]).

Let us also recall a fundamental result due to Nussbaum dealing with strongly commuting self-adjoint extensions of unbounded symmetric operators. For this we need to defined the notion of quasi-analytic vector for a given linear operator.

Definition 3.1.9.
Let T be a linear operator with $\mathcal{D}(T) \subset \mathcal{H}$. A vector $v \in \mathcal{D}^\infty(T) := \bigcap_{k=1}^{\infty} \mathcal{D}(T^k)$. is said to be quasi-analytic *for T if*

$$\sum_{k=1}^{\infty} \||T^k v\||^{-\frac{1}{k}} = \infty.$$

Theorem 3.1.10.
Let T_1 and T_2 be two unbounded symmetric operators with $\mathcal{D}(T_1)$ and $\mathcal{D}(T_2)$ subsets of the same Hilbert space $\mathcal{H}$. Let $\mathcal{D}$ be a set of vectors in $\mathcal{H}$ which are quasi-analytic for both T_1 and T_2 and such that $T_1\mathcal{D} \subset \mathcal{D}$, $T_2\mathcal{D} \subset \mathcal{D}$, $T_1T_2x = T_2T_1x$ for all $x \in \mathcal{D}$. If the set $\mathcal{D}$ is total in $\mathcal{H}$, i.e. $\overline{\operatorname{span}(\mathcal{D})} = \mathcal{H}$, then there exist unique self-adjoint extensions $\overline{T_1}$ and $\overline{T_2}$ of T_1 and T_2 in $\mathcal{H}$ such that $\overline{T_1}$ and $\overline{T_2}$ are strongly commuting.

(for a proof see e.g. [41, Theorem 6] and [49, Theorem 7.18]).

3.2 Solving the KMP for K compact semialgebraic sets

In Section 2.3 we proved the celebrated solution to the KMP for K compact due to Schmüdgen, see Corollary 2.3.17, by combining Schmüdgen Nichtnegativstellensatz and Riesz'-Haviland Theorem. In this section we are going to provide the original proof given by Schmüdgen in [48], which is based on an operator theoretical approach to the moment problem.

Theorem 3.2.1. *Let $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ linear and $S := \{g_1, \ldots, g_s\} \subset \mathbb{R}[\underline{X}]$ such that the associated bcsas K_S is compact. Then there exists a unique K_S-representing measure for L if and only if $L(h^2 g_1^{e_1} \cdots g_s^{e_s}) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$, $e_1, \ldots, e_s \in \{0, 1\}$.*

Proof.
Suppose there exists a K_S-representing measure μ for L, then for any $h \in \mathbb{R}[\underline{X}]$ and any $e_1, \ldots, e_s \in \{0,1\}$ we have

$$L(h^2 g_1^{e_1} \cdots g_s^{e_s}) = \int_{K_S} h^2 g_1^{e_1} \cdots g_s^{e_s} d\mu,$$

which is non-negative as integral of a non-negative function w.r.t. a non-negative measure.

Conversely, suppose that $L(h^2 g_1^{e_1} \cdots g_s^{e_s}) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$, $e_1, \ldots, e_s \in \{0,1\}$, i.e. $L(T_S) \subseteq [0, +\infty)$ where T_S is the preordering generated by S. We want to show the existence of a K_S-representing measure by using the Spectral Theorem 3.1.4.

First of all, let us observe that the compactness of K_S implies that there exists $\sigma > 0$ such that for any $x \in K_S$ we have $|x|^2 := x_1^2 + \cdots + x_n^2 < \sigma^2$, i.e. $\sigma^2 - |x|^2 > 0, \forall\, x \in K_S$. Hence, by Stengle Striktpositivstellensatz 1.3.1, we have that

$$\exists\, p, q \in T_S \text{ s.t. } (\sigma^2 - |x|^2)p = 1 + q. \tag{3.1}$$

Consider now the symmetric bilinear form

$$\begin{array}{rccl} \langle\ ,\ \rangle : & \mathbb{R}[\underline{X}] \times \mathbb{R}[\underline{X}] & \to & \mathbb{R} \\ & (\ p\ ,\ q\ ) & \mapsto & \langle p, q \rangle := L(pq) \end{array}$$

(note that $\langle \cdot, \cdot \rangle$ coincides with $\langle \cdot, \cdot \rangle_1$ as in Definition 2.3.9).

This is a quasi-inner product, since for any $f \in \mathbb{R}[\underline{X}]$ we have by assumption that $\langle f, f \rangle = L(f^2) \geq 0$ but $\langle f, f \rangle = 0$ does not necessarily imply that $f \equiv 0$ (e.g. if $L : \mathbb{R}[X] \to \mathbb{R}$ is linear s.t. $L(\underline{X}^n) = 1$ for $n = 0$ and $L(\underline{X}^n) = 0$ for $n \in \mathbb{N}$, then $\langle \underline{X}, \underline{X} \rangle = L(\underline{X}^2) = 0$ but $\underline{X}$ is not the zero polynomial.)

Let us consider the ideal $N := \{f \in \mathbb{R}[\underline{X}] : L(f^2) = 0\}$. Hence, there exists a well-defined inner product on the quotient vector space $\mathbb{R}[\underline{X}]/N$ which, by abuse of notation, we denote again by $\langle \cdot, \cdot \rangle$ and that is defined by

$$\langle f + N, r + N \rangle := L(fr), \forall f, r \in \mathbb{R}[\underline{X}]. \tag{3.2}$$

Let us denote by $\mathcal{H}_L$ the Hilbert space obtained by taking the completion of $\mathbb{R}[\underline{X}]/N$ w.r.t. the inner product $\langle \cdot, \cdot \rangle$ in (3.2)[2] and by $\| \cdot \|$ the norm on $\mathcal{H}_L$ induced by $\langle \cdot, \cdot \rangle$.

<u>*Claim*</u>: $\forall\ h \in \mathbb{R}[\underline{X}],\ j \in \{1, \ldots, n\},\ \ \|X_j h + N\| \leq \sigma \|h + N\|$.

[2]This construction is actually part of a very classical tool in operator theory named **<u>GNS-construction</u>** for Israel Gel'fand, Mark Naimark, and Irving Segal.

Proof. of Claim
Let us fix $h \in \mathbb{R}[\underline{X}]$ and $d \in \mathbb{N}$. Take p and q as in (3.1) and define $|\underline{X}|^2 := X_1^2 + \cdots + X_n^2$. Since $(1+q)|\underline{X}|^{2d-2}h^2 \in T_S$ and L is non-negative on elements of T_S, we have that:

$$\begin{aligned} L(|\underline{X}|^{2d}h^2p) &\leq L(|\underline{X}|^{2d}h^2p) + L\left((1+q)|\underline{X}|^{2d-2}h^2\right) \\ &= L\left(|\underline{X}|^{2d-2}h^2(|\underline{X}|^2p + 1 + q)\right) \\ &\overset{(3.1)}{=} L\left(|\underline{X}|^{2d-2}h^2\sigma^2p\right) = \sigma^2 L\left(|\underline{X}|^{2(d-1)}h^2p\right). \end{aligned}$$

Iterating, we get that

$$\forall\ d \in \mathbb{N},\ L(|\underline{X}|^{2d}h^2p) \leq \sigma^{2d}L(h^2p). \tag{3.3}$$

Fix $j \in \{1, \ldots, n\}$ and consider $\ell_j : \mathbb{R}[X_j] \to \mathbb{R}$ defined by $\ell_j(r) := L(rh^2)$, for all $r \in \mathbb{R}[X_j]$. Then ℓ_j is linear and $\ell_j(r^2) = L(r^2h^2) = L((rh)^2) \geq 0$, since by assumption L is non-negative on squares. Then, by Hamburger's Theorem 2.3.2 we have that there exists an $\mathbb{R}-$representing measure $\nu_{h,j}$ for ℓ_j. Therefore, for any $\lambda > 0$ and any $d \in \mathbb{N}$ we have

$$\begin{aligned} \int_{(-\infty,-\lambda)\cup(\lambda,+\infty)} \lambda^{2d} d\nu_{h,j} &\leq \int_{(-\infty,-\lambda)\cup(\lambda,+\infty)} X_j^{2d} d\nu_{h,j} \\ &\leq \int_{\mathbb{R}} X_j^{2d} d\nu_{h,j} = \ell_j(X_j^{2d}) = L(X_j^{2d}h^2) \\ &\leq L\left(X_j^{2d}h^2(|\underline{X}|^2p + 1 + q)\right) \\ &\overset{(3.1)}{=} L(X_j^{2d}h^2\sigma^2p) = \sigma^2 L(X_j^{2d}h^2p) \\ &\leq \sigma^2 L(|\underline{X}|^{2d}h^2p) \overset{(3.10)}{\leq} \sigma^{2+2d}L(h^2p). \end{aligned}$$

Hence, we proved that for any $\lambda > 0$ and any $d \in \mathbb{N}$ we have

$$\int_{(-\infty,-\lambda)\cup(\lambda,+\infty)} d\nu_{h,j} \leq \left(\frac{\sigma}{\lambda}\right)^{2d} \sigma^2 L(h^2p).$$

In particular, if we take $\lambda > \sigma$ and $d \to \infty$, then $\int_{(-\infty,-\lambda)\cup(\lambda,+\infty)} d\nu_{h,j} = 0$ and so that $\nu_{h,j}$ is supported in $[-\sigma, \sigma]$. Then

$$\begin{aligned} \|X_jh + N\|^2 &= L(X_j^2h^2) = \ell_j(X_j^2) = \int_{\mathbb{R}} X_j^2 d\nu_{h,j} = \int_{[-\sigma,\sigma]} X_j^2 d\nu_{h,j} \\ &\leq \sigma^2 \int_{[-\sigma,\sigma]} d\nu_{h,j} = \sigma^2\ell_j(1) = \sigma^2 L(h^2) = \sigma^2\|h + N\|^2. \end{aligned}$$

□(Claim)

For any $j \in \{1, \dots, n\}$, let us define the *multiplication operator* as follows

$$\begin{array}{rccc} W_j: & \mathbb{R}[\underline{X}]/N & \to & \mathbb{R}[\underline{X}]/N \\ & h+N & \mapsto & X_j h + N \end{array}$$

This is a well-defined operator with s.t. $\mathcal{D}(W_j) = \mathbb{R}[\underline{X}]/N$ is dense in $\mathcal{H}_L$ and

(a) $\underline{W_j \text{ is bounded}}$, since

$$\|W_j\|_{op} := \sup_{\substack{r\in\mathcal{D}(W_j)\\ r\neq o}} \frac{\|W_j r\|}{\|r\|} = \sup_{\substack{h\in\mathbb{R}[\underline{X}]\\ h\notin N}} \frac{\|X_j h + N\|}{\|h+N\|} \overset{\text{Claim}}{\leq} \sigma \sup_{\substack{h\in\mathbb{R}[\underline{X}]\\ h\notin N}} \frac{\|h+N\|}{\|h+N\|} = \sigma.$$

As $(\mathbb{R}[\underline{X}]/N, \|\cdot\|)$ is a normed space, this means that W_j is continuous.

(b) $\underline{W_j \text{ is symmetric}}$, since for any $h, r \in \mathbb{R}[\underline{X}]/N$ we have

$$\langle W_j h, r\rangle = L(X_j hr) = L(hX_j r) = \langle h, W_j r\rangle.$$

(c) $\underline{W_1, \dots, W_n \text{ are pairwise commuting}}$, since for any $j \neq k$ in $\{1, \dots, n\}$ and any $h \in \mathbb{R}[\underline{X}]$ we have

$$W_j W_k(h+N) = W_j(X_k h+N) = X_j X_k h+N = X_k X_j h+N = W_k W_j(h+N).$$

By Theorem 3.1.5 (applied for $Z = Y = \mathcal{H}_L$, $U = \mathbb{R}[\underline{X}]/N$, $\varphi = W_j$), there exists a unique bounded operator $\overline{W_j} : \mathcal{H}_L \to \mathcal{H}_L$ extending W_j and $\|\overline{W_j}\|_{op} = \|W_j\|_{op}$. Since each $\mathcal{D}(W_j)$ is dense in $\mathcal{H}_L$ and each W_j is bounded (so continuous), we have that properties (b) and (c) above hold also for $\overline{W_1}, \dots, \overline{W_n}$. Hence, $\overline{W_1}, \dots, \overline{W_n}$ are pairwise commuting bounded self-adjoint operators with $\mathcal{D}(\overline{W_j}) = \mathcal{H}_L$ for all $j \in \{1, \dots, n\}$. Then, by the Spectral Theorem 3.1.4, there exists a unique non-negative Radon measure μ such that

$$\langle (1+N), \overline{W_1}^{\alpha_1} \cdots \overline{W_n}^{\alpha_n}(1+N)\rangle = \int_{\mathbb{R}^n} \underline{X}^\alpha d\mu < \infty, \forall \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n \tag{3.4}$$

and μ is supported in $B_{\|\overline{W_1}\|_{op}}(0) \times \cdots \times B_{\|\overline{W_n}\|_{op}}(0) \overset{(a)}{\subseteq} [-\sigma, \sigma]^n =: Q$.

Since

$$\begin{aligned} \langle (1+N), \overline{W_1}^{\alpha_1} \cdots \overline{W_n}^{\alpha_n}(1+N)\rangle &= \langle (1+N), {W_1}^{\alpha_1} \cdots {W_n}^{\alpha_n}(1+N)\rangle \\ &= \langle (1+N), {X_1}^{\alpha_1} \cdots {X_n}^{\alpha_n} + N\rangle \\ &= L({X_1}^{\alpha_1} \cdots {X_n}^{\alpha_n}) = L(\underline{X}^\alpha), \end{aligned}$$

(3.4) becomes

$$L(\underline{X}^\alpha) = \int_{\mathbb{R}^n} \underline{X}^\alpha d\mu, \forall \alpha \in \mathbb{N}_0^n.$$

Hence, the spectral measure μ is a $Q-$representing measure for L. It remains to show that μ is actually supported on K_S.

For each $i \in \{1, \dots, n\}$ we have

$$0 \leq L(g_i h^2) = \int_Q g_i h^2 d\mu, \ \forall \ h \in \mathbb{R}[\underline{X}].$$

As Q is compact, we can apply the Stone-Weierstrass Theorem 2.3.27, we get

$$0 \leq \int_Q g_i f^2 d\mu, \ \forall \ f \in \mathcal{C}(Q).$$

Then

$$0 \leq \int_Q g_i f d\mu, \ \forall \ f \in \mathcal{C}(Q) \text{ s.t. } f \geq 0 \text{ on } Q$$

and so the linear functional

$$\begin{array}{rccl} \tilde{L}: & \mathcal{C}(Q) & \to & \mathbb{R} \\ & f & \mapsto & \int_Q g_i f d\mu \end{array}$$

is such that $\tilde{L}(f) \geq 0$ for all $f \geq 0$ on Q. Hence, by Riesz-Markov-Kakutani Theorem 2.2.5, there exists a unique non-negative Radon measure ν such that $\tilde{L}(f) = \int f d\nu$ for all $f \in \mathcal{C}(Q)$. But $\tilde{L}(f) = \int f g_i d\mu$ for all $f \in \mathcal{C}(Q)$, so the signed measure $g_i\mu$ must coincide with ν. Hence, $g_i\mu$ is a non-negative measure, which implies that the support of μ must be contained in the set of non-negativity of each g_i, i.e. μ is supported in K_S.

The uniqueness of the K_S-representing measure follows from Theorem 2.3.26 for $A = \mathbb{R}[\underline{X}]$ and $K = K_S$. □

The operator theoretical approach used in the proof of Theorem 3.2.1 can be also employed to provide an alternative proof to Corollary 2.3.18. This proof is indeed much closer to the original proof of this result due to Putinar [43].

Theorem 3.2.2. *Let $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ linear and $S := \{g_1, \dots, g_s\} \subset \mathbb{R}[\underline{X}]$ such that the quadratic module M_S generated by S is Archimedean. Then there exists a unique K_S-representing measure for L if and only if $L(h^2 g_i) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$ and $i \in \{0, 1, \dots, s\}$, where $g_0 := 1$.*

Proof. Suppose there exists a K_S-representing measure μ for L, then for any $h \in \mathbb{R}[\underline{X}]$ and any $i \in \{0, 1, \dots, s\}$ we have

$$L(h^2 g_i) = \int_{K_S} h^2 g_i d\mu,$$

which is non-negative as integral of a non-negative function w.r.t. a non-negative measure.

Conversely, suppose that $L(h^2 g_i)$ for all $h \in \mathbb{R}[\underline{X}]$ and all $i \in \{0, 1, \dots, s\}$, i.e. $L(M_S) \geq 0$. Since $g_0 := 1$, we have that $L(h^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$. Then we can run the GNS-construction as in the proof of Theorem 3.2.1 and construct the Hilbert space $\mathcal{H}_L$ associated to L by taking the completion of $\mathbb{R}[\underline{X}]/N$ w.r.t. the inner product $\langle \cdot, \cdot \rangle$ defined in (3.2), where $N := \{f \in \mathbb{R}[\underline{X}] : L(f^2) = 0\}$. Denote by $\| \cdot \|$ the norm on $\mathcal{H}_L$ induced by $\langle \cdot, \cdot \rangle$.

In the proof of Theorem 3.2.1 the compactness of K_S and the non-negativity of L on T_S implied the following bound

$$\forall\ h \in \mathbb{R}[\underline{X}],\ j \in \{1, \dots, n\},\quad \|X_j h + N\| \leq \sigma \|h + N\| \text{ for some } \sigma > 0, \quad (3.5)$$

which was fundamental in the rest of the proof. Here we still have compactness of K_S as M_S is Archimedean by Remark 1.3.32-c), but we have the non-negativity of L only on M_S which is contained in T_S. However, we can still derive (3.5) exploiting the Archimedeanity of M_S. Indeed, as M_S is Archimedean, for any $j \in \{1, \dots, n\}$ there exists $\lambda_j \in \mathbb{N}$ such that $\lambda_j \pm X_j^2 \in M_S$. This together with the non-negativity of L on M_S gives in particular that $L(h^2(\lambda_j - X_j^2)) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$. Hence, for each $j \in \{1, \dots, n\}$ and for each $h \in \mathbb{R}[\underline{X}]$, we obtain

$$\|X_j h + N\|^2 = L(X_j^2 h^2) \leq L(\lambda_j h^2) = \lambda_j L(h^2) \leq \sigma^2 \|h + N\|,$$

where $\sigma^2 := \max_{j=1,\dots,n} \lambda_j$. This proves that (3.5) holds and so we can continue the proof exactly as in the proof of Theorem 3.2.1 and show that there exists a K_S-representing measure. As for the uniqueness, we can apply also here Theorem 2.3.26 for $A = \mathbb{R}[\underline{X}]$ and $K = K_S$ since the Archimedeanity of M_S ensures that K_S is compact. □

3.3 Solving the KMP for K non-compact semialgebraic sets

Having in mind Theorem 3.2.1 and Theorem 3.2.2, it is natural to ask if the non-negativity of a linear functional on T_S or M_S is still sufficient to get the existence of a K_S-representing measure when K_S is not compact (and so M_S is not Archimedean). We already know that this is true for $K_S \subseteq \mathbb{R}$ with $S \supseteq S_{nat}$ by Corollary 2.3.1 (see also Theorems 2.3.2 and 2.3.3). But what about higher dimensions? In this section, we are going to see how the operator theoretical approach to the KMP sheds some light on this question.

A crucial role will be played by the following condition which will be further discussed in the next chapter.

Definition 3.3.1. *Given a sequence $m := (m_\alpha)_{\alpha\in\mathbb{N}_0^n}$ of non-negative real numbers, we say that m fulfills the* Carleman condition *if*

$$\sum_{k=1}^{\infty} m_{(0,\ldots 0,\underbrace{2k}_{j\text{-}th},0,\ldots,0)}^{-\frac{1}{2k}} = \infty, \quad \forall j\in\{1,\ldots,n\}. \tag{3.6}$$

Let us start by a result due to Nussbaum, who obtained in [41, Theorem 10] a solution to the KMP for $K=\mathbb{R}^n$ as a consequence of an important result concerning the theory of unbounded operators, namely Theorem 3.1.10. Indeed, in this case the multiplication operators defined in the previous section are not anymore guaranteed to be bounded, because we do not have either compactness or Archimedianity to ensure that the bound (3.5) holds. Hence, we need to deal with unbounded operators and use the results in Section 3.1.2.

Theorem 3.3.2.
Let $n\geq 2$ be an integer and $L:\mathbb{R}[X_1,\ldots,X_n]\to\mathbb{R}$ linear. If $L(h^2)\geq 0$ for all $h\in\mathbb{R}[X_1,\ldots,X_n]$ and fulfills the Carleman condition, i.e.

$$\sum_{k=1}^{\infty}\frac{1}{\sqrt[2k]{L(X_j^{2k})}}=\infty, \quad \forall j\in\{1,\ldots,n\}, \tag{3.7}$$

then there exists a unique $\mathbb{R}^n-$representing measure for L. Conversely, if there exists a unique $\mathbb{R}^n-$representing measure for L then $L(h^2)\geq 0$ for all $h\in\mathbb{R}[X_1,\ldots,X_n]$.

The existence part of this theorem is a higher dimensional version of Hamburger's theorem 2.3.2. We provide a proof just for the case $n=2$, since the proof structure for $n\geq 3$ is exactly the same. Afterwards, we will see how this proof can be adapted to the case $n=1$, giving an alternative proof to Hamburger's theorem 2.3.2.

Proof. of Existence in Theorem 3.3.2 for $n=2$.
Suppose there exists a $\mathbb{R}^2-$representing measure μ for L, then for any polynomial $h\in\mathbb{R}[X_1,X_2]=:\mathbb{R}[\underline{X}]$ we have $L(h^2)=\int_{\mathbb{R}^2}h^2d\mu$, which is non-negative as integral of a non-negative function w.r.t. a non-negative measure.

Conversely, suppose that $L(h^2)\geq 0$ for all $h\in\mathbb{R}[\underline{X}]$ and that the Carleman condition (3.7) holds. Then we can run the GNS-construction as in the previous section and construct the Hilbert space $\mathcal{H}_L$ associated to L by taking the completion of $\mathbb{R}[\underline{X}]/N$ w.r.t. the inner product $\langle\cdot,\cdot\rangle$ defined in (3.2), where $N := \{f \in \mathbb{R}[\underline{X}] : L(f^2) = 0\}$. Denote by $\|\cdot\|$ the norm on $\mathcal{H}_L$ induced by $\langle \cdot, \cdot \rangle$. For any $j \in \{1, 2\}$, let us define the *multiplication operator* as follows

$$\begin{array}{rccc} W_j : & \mathbb{R}[\underline{X}]/N & \to & \mathbb{R}[\underline{X}]/N \\ & h + N & \mapsto & X_j h + N \end{array}$$

This is a well-defined operator which is densely defined in $\mathcal{H}_L$ and symmetric, since $\mathcal{D}(W_j) = \mathbb{R}[\underline{X}]/N$ and

$$\langle W_j h, r \rangle = L(X_j h r) = L(h X_j r) = \langle h, W_j r \rangle, \quad \forall\, h, r \in \mathbb{R}[\underline{X}]/N.$$

Since the multiplication operators are unbounded, we aim to use the Spectral Theorem 3.1.8 and so we need to find pairwise strongly commuting self-adjoint extensions of the multiplication operators in $\mathcal{H}_L$. To this purpose, let us consider the set

$$\mathcal{D} := \{X_1^s X_2^t + N | s, t \in \mathbb{N}_0\}$$

and show that W_1, W_2 and $\mathcal{D}$ fulfill all the assumptions of Theorem 3.1.10.

a) $W_1\mathcal{D} \subset \mathcal{D}$ and $W_2\mathcal{D} \subset \mathcal{D}$ directly follow from the definitions of W_1, W_2 and $\mathcal{D}$.
b) For all $h \in \mathcal{D}$, say $h = X_1^s X_2^t + N$ for some $s, t \in \mathbb{N}_0$, we have

$$W_1 W_2(h + N) = X_1^{s+1} X_2^{t+1} + N = X_2^{t+1} X_1^{s+1} + N = W_2 W_1(h + N).$$

c) $\mathcal{D}$ is total in $\mathcal{H}_L$ since $\mathrm{span}(\mathcal{D}) = \mathbb{R}[\underline{X}]/N$ which is dense in $\mathcal{H}_L$ by construction.
d) *Claim*: Any $h \in \mathcal{D}$ is a quasi-analytic vector for both W_1 and W_2.

Then Theorem 3.1.10 guarantees that there exist unique self-adjoint extensions $\overline{W_1}$ and $\overline{W_2}$ of W_1 and W_2 in $\mathcal{H}_L$ s.t. $\overline{W_1}$ and $\overline{W_2}$ are strongly commuting. Moreover, $1 + N \in \mathcal{D}(W_1) = \mathcal{D}(W_2) = \mathbb{R}[\underline{X}]/N \subset \mathcal{H}_L$ is s.t. $\forall d \in \mathbb{N}_0$, $\forall\, i_1, \ldots, i_{d+1} \in \{1, 2\}$ we have

$$\overline{W_{i_d}} \cdot \overline{W_{i_{d-1}}} \cdots \overline{W_{i_1}}(1 + N) = X_{i_d} \cdots X_{i_1} + N \in \mathcal{D}(W_{i_{d+1}}) = \mathbb{R}[\underline{X}]/N.$$

Then we can apply the Spectral Theorem 3.1.8 to $\overline{W_1}$ and $\overline{W_2}$ and get that there exists a unique non-negative Radon measure μ on $\mathbb{R}^2$ such that

$$\langle (1+N), \underbrace{\overline{W_1} \cdots \overline{W_1}}_{\alpha_1 \text{ times}} \underbrace{\overline{W_2} \cdots \overline{W_2}}_{\alpha_2 \text{ times}} \cdot (1+N) \rangle = \int_{\mathbb{R}^2} X_1^{\alpha_1} X_2^{\alpha_2} d\mu(X_1, X_2),\ \forall\, \alpha_1, \alpha_2 \in \mathbb{N}_0. \tag{3.8}$$

Since

$$\begin{aligned}\langle(1+N), \overline{W_1}^{\alpha_1}\overline{W_2}^{\alpha_2}(1+N)\rangle &= \langle(1+N), W_1{}^{\alpha_1}W_2{}^{\alpha_2}(1+N)\rangle \\ &= \langle(1+N), X_1{}^{\alpha_1}X_2{}^{\alpha_2}+N\rangle \\ &= L(X_1{}^{\alpha_1}X_2{}^{\alpha_2}) = L(\underline{X}^\alpha),\end{aligned}$$

(3.8) becomes $L(\underline{X}^\alpha) = \int_{\mathbb{R}^2} \underline{X}^\alpha d\mu, \forall\alpha \in \mathbb{N}_0^2$. Hence, the spectral measure μ is an $\mathbb{R}^2-$representing measure for L. $\square$

Note that the fact that μ is the unique spectral measure coming from the unique self-adjoint extensions of the multiplication operators to $\mathcal{H}_L$ does not guarantee that μ is the unique $\mathbb{R}^n-$representing measure for L. Indeed, there could exist self-adjoint extension (resp. pairwise strongly commuting extensions) of the multiplication operators in another Hilbert space larger than $\mathcal{H}_L$ such that the corresponding spectral measure ν is also an $\mathbb{R}^n-$representing for L but clearly does not coincide with μ. Hence, we need an extra argument to show the uniqueness of the representing measure.

Before passing to the determinacy part, let us complete the existence part by showing that the Claim d) holds. To do that we will need the notion of log-convex sequences and some of their properties.

Definition 3.3.3.
A sequence $(s_k)_{k\in\mathbb{N}_0}$ of non-negative real numbers is said to be log-convex *if for all $k \in \mathbb{N}$ we have that $s_k^2 \leq s_{k-1}s_{k+1}$.*

Lemma 3.3.4. *A sequence $(s_k)_{k\in\mathbb{N}_0}$ of positive real numbers is log-convex if and only if $\left(\sqrt[k]{\frac{s_k}{s_0}}\right)_{k\in\mathbb{N}}$ is monotone increasing.*

Proof.
The log-convexity of $(s_k)_{k\in\mathbb{N}_0}$ is equivalent to the sequence $\left(\frac{s_k}{s_{k-1}}\right)_{k\in\mathbb{N}}$ being increasing, since for any $k \in \mathbb{N}$ we have that

$$s_k^2 \leq s_{k-1}s_{k+1} \Leftrightarrow \frac{s_k}{s_{k-1}} \leq \frac{s_{k+1}}{s_k}.$$

Hence, for any $k \in \mathbb{N}$ we get

$$\frac{s_k}{s_0} = \prod_{j=1}^{k} \frac{s_j}{s_{j-1}} \leq \left(\frac{s_k}{s_{k-1}}\right)^k,$$

i.e. $s_{k-1}^k \leq s_0 s_k^{k-1}$. By multiplying the latter on both sides by $\frac{1}{s_0^k}$ we get $\left(\frac{s_{k-1}}{s_0}\right)^k \leq \left(\frac{s_k}{s_0}\right)^{k-1}$, which is equivalent to $\left(\frac{s_{k-1}}{s_0}\right)^{\frac{1}{k-1}} \leq \left(\frac{s_k}{s_0}\right)^{\frac{1}{k}}$. $\square$

Lemma 3.3.5. *Let $(s_k)_{k\in\mathbb{N}_0}$ be a sequence of non-negative real numbers s.t. $s_{2k}>0$ for all $k\in\mathbb{N}_0$ and $(s_{2k})_{k\in\mathbb{N}_0}$ is log-convex. Then*

$$\sum_{k=1}^{\infty}\frac{1}{\sqrt[2k]{s_{2k}}}=\infty \Longleftrightarrow \sum_{k=1}^{\infty}\frac{1}{\sqrt[4k]{s_{4k}}}=\infty.$$

Proof. (see Bonus Sheet) □

Lemma 3.3.6.
Let $o\neq q,f\in\mathbb{R}[\underline{X}]$ and $L:\mathbb{R}[\underline{X}]\to\mathbb{R}$ linear s.t. $L(h^2)\geq 0$ for all $h\in\mathbb{R}[\underline{X}]$. Define $\tilde{L}_f:\mathbb{R}[\underline{X}]\to\mathbb{R}$ as $\tilde{L}_f(p):=L(fp)$ for all $p\in\mathbb{R}[\underline{X}]$. Then

$$\left(\tilde{L}_f(h^2)\geq 0,\ \forall h\in\mathbb{R}[\underline{X}] \text{ and } \sum_{k=1}^{\infty}\frac{1}{\sqrt[2k]{L(q^{2k})}}=\infty\right)\Longrightarrow\sum_{k=1}^{\infty}\frac{1}{\sqrt[2k]{\tilde{L}_f(q^{2k})}}=\infty.$$

Proof.
For any $k\in\mathbb{N}_0$, set $t_k:=L(q^k)$ and $r_k:=\tilde{L}_f(q^k)$. Since $L(h^2)\geq 0$ and $\tilde{L}_f(h^2)\geq 0$ for all $h\in\mathbb{R}[\underline{X}]$, we have that $t_{2k}\geq 0$, $r_{2k}\geq 0$ for all $k\in\mathbb{N}_0$ and we can apply the Cauchy-Schwarz inequality to both L and $\tilde{L}_f$. Hence, we obtain that the following hold for all $k\in\mathbb{N}_0$

$$t_{2k+2}^2=\left(L(q^{2k+2})\right)^2=\left(L(q^kq^{k+2})\right)^2\leq L(q^{2k})L(q^{2k+4})=t_{2k}t_{2k+4} \quad (3.9)$$

$$r_{2k}^2=\left(L(fq^{2k})\right)^2\leq L(q^{4k})L(f^2)=t_{4k}L(f^2) \quad (3.10)$$

Now w.l.o.g. we can assume that $t_{2k}>0$ for all $k\in\mathbb{N}_0$ and $L(f^2)>0$. Indeed,

- If $t_{2j}=0$ for some $j\in\mathbb{N}_0$, then by (3.9) we have that $t_{2k}=0$ for all $k\geq j$ in $\mathbb{N}_0$ and so by (3.10) also $r_{2k}=0$ for all $k\geq j$ in $\mathbb{N}_0$. Hence, $\sum_{k=1}^{\infty}\frac{1}{\sqrt[2k]{r_{2k}}}=\infty$ and we have already our desired conclusion.
- If $L(f^2)=0$, then $r_{2k}=0$ for all $k\in\mathbb{N}_0$ and so again our desired conclusion holds.

Hence, $(t_{2k})_{k\in\mathbb{N}_0}$ is a sequence of positive real numbers, which is log-convex by (3.9). Since by assumption $\sum_{k=1}^{\infty}\frac{1}{\sqrt[2k]{t_{2k}}}=\infty$, we can apply Lemma 3.3.5 and obtain that

$$\sum_{k=1}^{\infty}\frac{1}{\sqrt[4k]{t_{4k}}}=\infty \quad (3.11)$$

Therefore, we get

$$r_{2k}^{-\frac{1}{2k}}\overset{(3.10)}{\geq} t_{4k}^{-\frac{1}{4k}}\left(L(f^2)\right)^{-\frac{1}{4k}}\geq c_f t_{4k}^{-\frac{1}{4k}},\ \forall\, k\in\mathbb{N}, \quad (3.12)$$

where $c_f := \left(1 + L(f^2)\right)^{-1}$ is clearly a positive constant. Then

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{r_{2k}}} \overset{(3.12)}{\geq} c_f \sum_{k=1}^{\infty} \frac{1}{\sqrt[4k]{t_{4k}}} \overset{(3.11)}{=} \infty.$$

□

Corollary 3.3.7. *Let $o \neq f \in \mathbb{R}[\underline{X}]$ and $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ linear s.t. $L(h^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$. Suppose that $\tilde{L}_f(h^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$. If L fulfills Carleman condition (3.7), then so does $\tilde{L}_f$.*

Proof. Apply Lemma 3.3.6 for $q = X_j$ for each $j \in \{1, \dots, n\}$. □

Proof. of Claim d).
Let us fix $s, t \in \mathbb{N}_0$, then by using the Cauchy-Schwarz inequality we get that for any $k \in \mathbb{N}$

$$\left\| {W_1}^k X_1^s X_2^t \right\|^2 = \left(L(X_1^{2(k+s)} X_2^{2t}) \right) \leq \left(L(X_1^{4(k+s)}) \right)^{\frac{1}{2}} \left(L(X_2^{4t}) \right)^{\frac{1}{2}}$$

which gives in turn that

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{\left\| {W_1}^k X_1^s X_2^t \right\|}} \geq \sum_{k=1}^{\infty} \frac{1}{\sqrt[4k]{L(X_1^{4(k+s)}) L(X_2^{4t})}}. \tag{3.13}$$

W.l.o.g. we can assume that $c := L(X_2^{4t}) > 0$ and that for any $k \in \mathbb{N}$ we have $L(X_1^{4(k+s)}) > 0$ (otherwise the series on the right-hand side of (3.13) would diverge and we would have already our conclusion).

Then $L(cX_1^{4s} h^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$. This together with (3.7) ensures that we can apply Lemma 3.3.6 for $q := X_1$ and $f := cX_1^{4s}$ obtaining that $\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{cL(X_1^{2k+4s})}} = \infty$.

Moreover, the sequence $\left(L(cX_1^{2k+4s}) \right)_{k \in \mathbb{N}_0}$ is log-convex (see Definition 3.3.3), since for any $k \in \mathbb{N}$ we have

$$\left[L(cX_1^{2k+4s}) \right]^2 = \left[L(\sqrt{c} X_1^{k-1+2s} \sqrt{c} X_1^{k+1+2s}) \right]^2 \leq L(cX_1^{(2k-2)+4s}) L(cX_1^{(2k+2)+4s}).$$

Then we have that

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[4k]{cL(X_1^{4k+4s})}} = \infty. \tag{3.14}$$

In fact, we can distinguish two cases:

- If there exists $w \in \mathbb{N}_0$ such that $L(cX_1^{2w+4s}) = 0$, then by log-convexity $L(cX_1^{2k+4s}) = 0$ for all integers $k \geq w$, which implies that (3.14) holds.
- if $L(cX_1^{2k+4s}) > 0$ for all $k \in \mathbb{N}$, then by Lemma 3.3.5 we have that (3.14) holds.

Hence, (3.14) and (3.13) guarantee that $\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{\| {W_1}^k X_1^s X_2^t \|}} = \infty$, i.e. $X_1^s X_2^t$ is a quasi-analytic vector for W_1.

The same proof applies to show that $X_1^s X_2^t$ is a quasi-analytic vector for W_2. □(Claim d))

The proof of the existence part of Theorem 3.3.2 can be adapted to provide an alternative proof to Hamburger's theorem 2.3.2. Note that for $n = 1$ the Carleman condition is not needed for getting the existence of an $\mathbb{R}^n-$representing measure for L, while this was essential for getting it in the case $n \geq 2$. We will see that Carleman's condition is instead crucial in proving the determinacy of the $\mathbb{R}^n-$representing measure independently of the dimension n.

Theorem 3.3.8. *Let $L : \mathbb{R}[X] \to \mathbb{R}$ be linear. There exists an $\mathbb{R}-$representing measure for L if and only if $L(h^2) \geq 0$ for all $h \in \mathbb{R}[X]$.*
If in addition, L fulfills the Carleman condition (3.7) for $n = 1$, i.e.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{L(X^{2k})}} = \infty. \tag{3.15}$$

then the representing measure is determinate.

Proof. of Existence in Theorem 3.3.8, i.e. of Hamburger's theorem 2.3.2
Suppose there exists an $\mathbb{R}-$representing measure μ for L, then for any polynomial $h \in \mathbb{R}[X]$ we have $L(h^2) = \int_{\mathbb{R}} h^2 d\mu$, which is non-negative as integral of a non-negative function w.r.t. a non-negative measure.

Conversely, suppose that $L(h^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$. Then we can run the GNS-construction and construct the Hilbert space $\mathcal{H}_L$ associated to L. Consider the multiplication operator

$$\begin{array}{rccc} W: & \mathbb{R}[X]/N & \to & \mathbb{R}[X]/N \\ & h+N & \mapsto & X_j h + N \end{array}$$

where $N := \{f \in \mathbb{R}[X] : L(f^2) = 0\}$. Since W is a symmetric unbounded operator densely defined in $\mathcal{H}_L$, it admits a self-adjoint extension $\overline{W}$ in $\mathcal{H}_L$ (see e.g. [45, p.319]). Then by the Spectral Theorem 3.1.8 for $n = 1$ and

$v = 1 + N \in \mathcal{D}^\infty(W) = \mathbb{R}[\underline{X}]/N \subset \mathcal{H}_L$ we get that there exists a unique non-negative Radon measure μ on $\mathbb{R}$ such that

$$\langle (1+N), \overline{W}^j(1+N)\rangle = \int_{\mathbb{R}} X^j d\mu(X), \ \forall \ j \in \mathbb{N}_0. \tag{3.16}$$

Since

$$\langle (1+N), \overline{W}^j(1+N)\rangle = \langle (1+N), W^j(1+N)\rangle = \langle (1+N), X^j + N\rangle = L(X^j)$$

(3.16) becomes $L(X) = \int_{\mathbb{R}} X^j d\mu(X), \forall \ j \in \mathbb{N}_0$. Hence, the spectral measure μ is an $\mathbb{R}-$representing measure for L. □

Let us show now the determinacy part of both Theorem 3.3.2 and Theorem 3.3.8. This will be a consequence of the following important result about the determinacy of the moment problem, which we are going to prove in the next chapter.

Theorem 3.3.9.
Let $n \in \mathbb{N}$. If μ is a non-negative Radon measure on $\mathbb{R}^n$ such that the sequence of its moments $(m_\alpha^\mu)_{\alpha\in\mathbb{N}_0^n}$ exists and fulfills the Carleman condition (3.6), then μ is determinate, i.e. any other non-negative Radon measure having the same moment sequence as μ must coincide with μ.

Proof. (of Uniqueness in Theorem 3.3.2 and in Theorem 3.3.8)
Let μ, ν be two $\mathbb{R}^n-$representing measure for L. Then μ and ν have the same moment sequence $(L(\underline{X}^\alpha))_{\alpha\in\mathbb{N}_0^n}$. Since by assumption L fulfills (3.7), the sequence $(L(\underline{X}^\alpha))_{\alpha\in\mathbb{N}_0^n}$ fulfills (3.6) and so Theorem 3.3.9 ensures that $\mu = \nu$. □

Carleman's condition, and so Theorem 3.3.9, will also play a crucial role to prove a version of Theorem 3.3.2 for the KMP with K (not necessarily compact) b.c.s.a.s. of $\mathbb{R}^n$ due to Lasserre [33, Theorem 3.2] (see also [23, Theorem 5.1]).

Theorem 3.3.10.
Let $n, s \in \mathbb{N}$, $S := \{g_1, \ldots, g_s\} \subset \mathbb{R}[X_1, \ldots, X_n]$, and $L : \mathbb{R}[X_1, \ldots, X_n] \to \mathbb{R}$ linear s.t. $L(h^2) \geq 0$ for all $h \in \mathbb{R}[X_1, \ldots, X_n]$ and Carleman's condition (3.7) holds. Then there exists a unique K_S-representing measure for L if and only if $L(g_i h^2) \geq 0$ for all $h \in \mathbb{R}[X_1, \ldots, X_n]$ and all $i \in \{1, \ldots, s\}$.

Proof.
Since $L(h^2) \geq 0$ for all $h \in \mathbb{R}[X_1, \dots, X_n] =: \mathbb{R}[\underline{X}]$ and L fulfills the Carleman condition (3.7), Theorem 3.3.2 guarantees that there exists a unique $\mathbb{R}^n-$representing measure μ for L. We want to show that μ is actually supported on K_S.
Case $s = 1$
For notational convenience, let us first consider the case $s = 1$ and so $S := \{g\}$. Define $\tilde{L}_g : \mathbb{R}[\underline{X}] \to \mathbb{R}$ as $\tilde{L}_g(p) := L(pg)$ for all $p \in \mathbb{R}[\underline{X}]$. Since $\tilde{L}_g(h^2) = L(gh^2) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$ and L satisfies the Carleman condition (3.7), Lemma 3.3.6 (applied for $q = X_j$ with $j = 1, \dots, n$ and $f = g$) ensures that $\tilde{L}_g$ also fulfils Carleman's condition. Hence, by applying again Theorem 3.3.2 we get that there exists a unique $\mathbb{R}^n-$representing measure η for $\tilde{L}_g$. Thus, we obtained that

$$\int_{\mathbb{R}^n} \underline{X}^\alpha d\eta(\underline{X}) = \tilde{L}_g(\underline{X}^\alpha) = L(g\underline{X}^\alpha) = \int_{\mathbb{R}^n} \underline{X}^\alpha \underbrace{g(\underline{X})d\mu(\underline{X})}_{=:d\nu(\underline{X})}, \ \forall\, \alpha \in \mathbb{N}_0^n. \quad (3.17)$$

The measure ν is a signed Radon measure on the Borel $\sigma-$algebra $\mathcal{B}(\mathbb{R}^n)$ on $\mathbb{R}^n$ and can be written as $\nu = \nu_+ - \nu_-$, where

$$\begin{aligned} d\nu_+ &:= \ \mathbb{1}_{\Gamma^+} d\nu \quad \text{with} \quad \Gamma^+ := \{x \in \mathbb{R}^n : g(x) \geq 0\} \\ d\nu_- &:= -\mathbb{1}_{\Gamma^-} d\nu \quad \text{with} \quad \Gamma^- := \{x \in \mathbb{R}^n : g(x) < 0\} \end{aligned}$$

and so ν_+ and ν_- are both non-negative Radon measures on $\mathbb{R}^n$.

Claim: $\nu_- \equiv 0$.

Proof.
Define the following two non-negative Radon measures on $\mathcal{B}(\mathbb{R}^n)$

$$d\mu_+ := \mathbb{1}_{\Gamma^+} d\mu \text{ and } d\mu_- := \mathbb{1}_{\Gamma^-} d\mu.$$

Then $\mu = \mu_+ + \mu-$ and so we have

$$\int_{\mathbb{R}^n} X_j^{2k} d\mu_+(\underline{X}) \leq \int_{\mathbb{R}^n} X_j^{2k} d\mu(\underline{X}), \ \forall\, k \in \mathbb{N}_0, \forall j = 1, \dots, n. \quad (3.18)$$

Consider $\ell_{\mu_+} : \mathbb{R}[\underline{X}] \to \mathbb{R}$ defined by $\ell_{\mu_+}(p) := \int_{\mathbb{R}^n} p d\mu_+$. Then (3.18) can be rewritten as

$$\ell_{\mu_+}(X_i^{2k}) \leq L(X_j^{2k}), \ \ \forall\, k \in \mathbb{N}_0, \forall j = 1, \dots, n,$$

which implies that

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{\ell_{\mu_+}(X_i^{2k})}} \geq \sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{L(X_j^{2k})}} \stackrel{\text{hp}}{=} \infty, \ \forall\, k \in \mathbb{N}_0, \forall j = 1, \dots, n,$$

i.e. ℓ_{μ_+} fulfills the Carleman condition.

Consider $\ell_{\nu_+} : \mathbb{R}[\underline{X}] \to \mathbb{R}$ defined by $\ell_{\nu_+}(p) := \int_{\mathbb{R}^n} p d\nu_+$. Then

$$\ell_{\nu_+}(p) = \int_{\mathbb{R}^n} p \mathbb{1}_{\Gamma^+} g d\mu = \int_{\mathbb{R}^n} pg d\mu_+ = \ell_{\mu_+}(pg), \ \forall p \in \mathbb{R}[\underline{X}]$$

and

$$\ell_{\nu_+}(h^2) = \int_{\mathbb{R}^n} h^2 d\nu_+ \geq 0 \ \forall h \in \mathbb{R}[\underline{X}].$$

Hence, by Lemma 3.3.6 (applied for $L = \ell_{\mu_+}$, $q = X_j$,$f = g$), we get that also ℓ_{ν_+} fulfills the Carleman condition and so that ν_+ is determinate by Theorem 3.3.9.

Putting all together, we obtain that for all $\alpha \in \mathbb{N}_0^n$

$$\begin{aligned}
\int_{\mathbb{R}^n} \underline{X}^\alpha d\nu_+(\underline{X}) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \underline{X}^\alpha g(\underline{X}) d\mu_+(\underline{X}) \\
&\stackrel{\mu=\mu_+\mu_-}{=} \int_{\mathbb{R}^n} \underline{X}^\alpha g(\underline{X}) d\mu(\underline{X}) - \int_{\mathbb{R}^n} \underline{X}^\alpha g(\underline{X}) d\mu_-(\underline{X}) \\
&\stackrel{(3.17)}{=} \int_{\mathbb{R}^n} \underline{X}^\alpha d\eta(\underline{X}) - \int_{\mathbb{R}^n} \underline{X}^\alpha g(\underline{X}) d\mu_-(\underline{X}) \\
&\stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \underline{X}^\alpha d\eta(\underline{X}) + \int_{\mathbb{R}^n} \underline{X}^\alpha d\nu_-(\underline{X}) \\
&= \int_{\mathbb{R}^n} \underline{X}^\alpha d(\eta + \nu_-)(\underline{X}),
\end{aligned}$$

i.e. the non-negative Radon measures ν_+ and $\eta + \nu_-$ have the same moments. Since ν_+ is determinate, they need to coincide, i.e. $\nu_+ \equiv \eta + \nu_-$. Hence, for any $B \in \mathcal{B}(\mathbb{R}^n)$ we have $0 = \nu_+(\Gamma^-) \geq \nu_-(\Gamma^-) \geq 0$, that is, $\nu_-(\Gamma^-) = 0$. Since by definition $\nu_-(\Gamma^+) = 0$ and $\mathbb{R}^n = \Gamma^+ \cup \Gamma^-$, we get that $\nu_- \equiv 0$. $\square$(Claim)

The Claim implies that μ is supported on Γ^+, i.e. for any $B \in \mathcal{B}(\mathbb{R}^n)$ such that $B \cap \Gamma^+ = \emptyset$ we have $\mu(B) = 0$. In fact, suppose that this is not the case. Then there exists $\varepsilon > 0$ such that $\overline{B}_\varepsilon \cap \Gamma^+ = \emptyset$ but $\mu\left(\overline{B}_\varepsilon\right) > 0$, where $\overline{B}_\varepsilon$ is some closed ball in $\mathbb{R}^n$ of radius ε. Then for any $x \in \overline{B}_\varepsilon$ we have that $x \in \Gamma^-$ and so $g(x) < 0$, i.e. $-g(x) > 0$. Hence, we get

$$0 \stackrel{\text{Claim}}{=} \nu_-(\overline{B}_\varepsilon) = \int_{\overline{B}_\varepsilon} -\mathbb{1}_{\Gamma^-} d\nu = \int_{\overline{B}_\varepsilon} -g(\underline{X}) d\mu(\underline{X}) \geq \left(\min_{x \in \overline{B}_\varepsilon} -g(x)\right) \mu(\overline{B}_\varepsilon) > 0,$$

which yields a contradiction.

Thus, we proved that μ is supported on $\{x \in \mathbb{R}^n : g(x) \geq 0\}$, which in this case coincides with K_S.

Case $s \geq 2$

Suppose now that $s > 1$ and $S := \{g_1, \dots, g_s\}$. By repeating for each g_i the same proof as above, we get that μ is supported on each $\{x \in \mathbb{R}^n : g_i(x) \geq 0\}$ with $i \in \{1, \dots, s\}$. Hence, we get that

$$\begin{aligned} 0 \leq \mu\left(\mathbb{R}^n \setminus K_S\right) &= \mu\left(\bigcup_{i=1}^{s} \mathbb{R}^n \setminus \{x \in \mathbb{R}^n : g_i(x) \geq 0\}\right) \\ &\leq \sum_{i=1}^{s} \mu\left(\mathbb{R}^n \setminus \{x \in \mathbb{R}^n : g_i(x) \geq 0\}\right) = 0, \end{aligned}$$

i.e. μ is supported on K_S.

□

Chapter 4

Determinacy of the $K-$Moment Problem

In this chapter we are going to investigate the so-called *determinacy question*, which is certainly one of the most investigated aspects of the $K-$moment problem. The determinacy question consists in finding under which conditions a non-negative measure with given support K is completely determined by its moments. In particular, we will see how the concept of quasi-analyticity enters in the study of the determinacy question and give a proof of Theorem 3.3.9 first for $n = 1$ and then for higher dimensions.

From now on, for $K \subseteq \mathbb{R}^n$ closed, we denote by $\mathcal{M}^*(K)$ the collection of all the non-negative Radon measures on $\mathbb{R}^n$ having finite moments of all orders and which are supported in K.

Definition 4.0.1. *A measure $\mu \in \mathcal{M}^*(K)$ is said to be $K-$*determinate *if for any $\nu \in \mathcal{M}^*(K)$ such that $\int x^\alpha d\mu(x) = \int x^\alpha d\nu(x), \forall\ \alpha \in \mathbb{N}_0^n$ we have that $\mu \equiv \nu$. Equivalently a sequence of real numbers m (resp. a linear functional L on $\mathbb{R}[\underline{X}]$) is called $K-$determinate if there exists at most one $K-$representing measure for m (resp. for L).*

Note that if K_1 and K_2 are closed subsets of $\mathbb{R}^n$ such that $K_1 \subset K_2$, then the K_2-determinacy always implies the K_1-determinacy but the converse does not hold in general.

4.1 Quasi-analytic classes

Let us recall the basic definitions and state some preliminary results concerning the theory of quasi-analytic functions. In the following, we denote by $\mathcal{C}^\infty(X)$ the space of all infinitely differentiable real valued functions defined on a topological space X.

Definition 4.1.1.
Given a sequence of positive real numbers $(s_j)_{j\in\mathbb{N}_0}$ and an open $I \subseteq \mathbb{R}$, we define the class $C\{s_j\}$ as the set of all functions $f \in \mathcal{C}^\infty(I)$ for which there exists $\gamma_f > 0$ (only depending on f) such that $\left\|D^k f\right\|_\infty \leq (\gamma_f)^k s_k, \quad \forall\, k \in \mathbb{N}_0$, where $D^k f$ is the $k-$th derivative of f and $\left\|D^k f\right\|_\infty := \sup_{x\in I}\left|D^k f(x)\right|$.

The class $C\{s_j\}$ of functions on I is said to be quasi-analytic *if the conditions*

$$f \in C\{s_j\},\ \exists\ t_0 \in I s.t.\ \ (D^k f)(t_0) = 0, \quad \forall\, k \in \mathbb{N}_0$$

imply that $f(x) = 0$ for all $x \in I$.

The problem to give necessary and sufficient conditions bearing on the sequence $(s_j)_{j\in\mathbb{N}_0}$ such that the class $C\{s_j\}$ is quasi-analytic was proposed by Hadamard in [17]. Denjoy was the first to provide sufficient conditions for the quasi-analyticity of a class [10], but the problem was completely solved by Carleman, who generalized Denjoy's theorem and methods giving the first characterization of quasi-analytic classes in [6].

Theorem 4.1.2 (The Denjoy-Carleman Theorem)**.**
Let $(s_j)_{j\in\mathbb{N}_0}$ be a sequence of positive real numbers. The class $C\{s_k\}$ is quasi-analytic if and only if

$$\sum_{k=1}^{\infty} \frac{1}{\inf_{j\geq k} \sqrt[j]{s_j}} = \infty.$$

Proof. see e.g. [8] for a simple but detailed proof. □

Corollary 4.1.3. *If $(s_j)_{j\in\mathbb{N}_0}$ is a sequence of positive real numbers such that*

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{s_k}} = \infty,$$

then the class $C\{s_j\}$ is quasi-analytic.

Proof. For any $k \in \mathbb{N}$ we have $\inf_{j\geq k} \sqrt[j]{s_j} \leq \sqrt[k]{s_k}$ and so

$$\sum_{k=1}^{\infty} \frac{1}{\inf_{j\geq k} \sqrt[j]{s_j}} \geq \sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{s_k}}.$$

Since by assumption the series on right-hand side diverges, so does the series on the left-hand side. Hence, by Theorem 4.1.2, the class $C\{s_j\}$ is quasi-analytic. □

Remark 4.1.4. *If $(s_j)_{j\in\mathbb{N}_0}$ is a log-convex sequence of positive real numbers such that $s_0 = 1$, then in Corollary 4.1.3 also the converse implication holds. Indeed, under these assumptions the sequence $(\sqrt[j]{s_j})_{j\in\mathbb{N}}$ is increasing by Lemma 3.3.4 and so for each $k \in \mathbb{N}$ we have $\inf_{j\geq k} \sqrt[j]{s_j} = \sqrt[k]{s_k}$.Hence, the condition $\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{s_k}} = \infty$ is equivalent to $\sum_{k=1}^{\infty} \frac{1}{\inf_{j\geq k} \sqrt[j]{s_j}} = \infty$ and so to the quasi-analiticity of the class $C\{s_j\}$ by Theorem 4.1.2.*

Using Corollary 4.1.3, we can easily produce some examples of quasi-analytic classes.

Examples 4.1.5.

- *The class $\mathcal{C}\{j^j\}$ is quasi-analytic, since $\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{k^k}} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$.*
- *The class $\mathcal{C}\{j!\}$ is quasi-analytic, since $\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{k!}} \geq \sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{k^k}} = \infty$. This is in fact the class of real analytic functions. Recall that a function f is real analytic on $I \subseteq \mathbb{R}$ if $f \in \mathcal{C}^\infty(I)$ and the Taylor series of f at any point $x_0 \in I$ pointwise converges to f in a neighborhood of x_0.*

4.2 Determinacy in the one dimensional case

In this section we are going to exploit the theory of quasi-analytic functions on $\mathbb{R}$ to prove the so-called *Carleman's Theorem*, i.e. Theorem 3.3.9 for $n = 1$. Carleman was indeed the first to approach the determinacy question with methods involving quasi-analyticity theory in his famous work of 1926 (see [6, Chapter VIII]).

Theorem 4.2.1 (Carleman's Theorem)**.**
If $\mu \in \mathcal{M}^*(\mathbb{R})$ is such that its moment sequence $(m_j^\mu)_{j\in\mathbb{N}_0}$ fulfils the following

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2k]{m_{2k}^\mu}} = \infty, \tag{4.1}$$

then μ is $\mathbb{R}-$determinate.

The original proof by Carleman makes use of the Cauchy transform of the given measure. Here, we propose a slightly different proof that uses the Fourier-Stieltjes transform but maintains the same spirit of Carleman's proof. Before proving Theorem 4.2.1, let us recall the definition of Fourier-Stieltjes transform of a measure and some fundamental properties of this object.

Definition 4.2.2. *Let $\mu \in \mathcal{M}^*(\mathbb{R})$. The Fourier-Stieltjes transform of μ is the function $F_\mu \in \mathcal{C}^\infty(\mathbb{R})$ defined by*

$$F_\mu(t) := \int_{\mathbb{R}} e^{-ixt} d\mu(x), \forall\ t \in \mathbb{R}.$$

Proposition 4.2.3. *Let $\mu, \nu \in \mathcal{M}^*(\mathbb{R})$.*
a) If $F_\mu \equiv F_\nu$ on $\mathbb{R}$, then $\mu \equiv \nu$.
b) For any $k \in \mathbb{N}_0$ and any $t \in \mathbb{R}$, we have $(D^k F_\mu)(t) = \int_{\mathbb{R}} (-ix)^k e^{-ixt} d\mu(x)$.

Proof. of Theorem 4.2.1
W.l.o.g. assume that all even moments of μ are positive. In fact, if $m_{2j}^\mu = 0$ for some $j \in \mathbb{N}_0$, then μ is supported in $\{x \in \mathbb{R} : x^{2j} = 0\} = \{0\}$ and thus, $\mu = m_0^\mu \delta_{\{0\}}$ is the unique measure having these moments, which proves already the determinacy of μ.

Let $\nu \in \mathcal{M}^*(\mathbb{R})$ having the same moment sequence as μ and let us consider the Fourier-Stieltjes transforms of μ and ν. Then $(F_\mu - F_\nu) \in \mathcal{C}^\infty(\mathbb{R})$ and for any $k \in \mathbb{N}_0$ and any $t \in \mathbb{R}$ we get

$$(D^k(F_\mu - F_\nu))(t) = \int_{\mathbb{R}} (-ix)^k e^{-ixt} \mu(dx) - \int_{\mathbb{R}} (-ix)^k e^{-ixt} \nu(dx) \tag{4.2}$$

and so

$$\begin{aligned} \left|(D^k(F_\mu(t) - F_\nu))(t)\right| &\leq \int_{\mathbb{R}} |x|^k \mu(dx) + \int_{\mathbb{R}} |x|^k \nu(dx) \\ &\overset{\text{Hölder}}{\leq} \sqrt{m_0^\mu m_{2k}^\mu} + \sqrt{m_0^\nu m_{2k}^\nu} \\ &= 2\sqrt{m_0^\mu m_{2k}^\mu} \leq (1+\gamma)\sqrt{m_{2k}^\mu}, \end{aligned}$$

where $\gamma := 2\sqrt{m_0^\mu} > 0$. Hence, $F_\mu - F_\nu \in \mathcal{C}\{s_k\}$, where $s_k := (1+\gamma)\sqrt{m_{2k}^\mu}$ for any $k \in \mathbb{N}_0$.

Since

$$\sum_{k=1}^\infty \frac{1}{\sqrt[k]{s_k}} = \sum_{k=1}^\infty \frac{1}{\sqrt[k]{(1+\gamma)\sqrt{m_{2k}^\mu}}} \geq \frac{1}{(1+\gamma)} \sum_{k=1}^\infty \frac{1}{\sqrt[2k]{m_{2k}^\mu}} \overset{(4.1)}{=} \infty,$$

Corollary 4.1.3 guarantees that the class $\mathcal{C}\{s_k\}$ is quasi-analytic.

Moreover, (4.2) gives in particular $(D^k(F_\mu - F_\nu))(0) = 0$ for all $k \in \mathbb{N}_0$. Then the quasi-analyticity of the class $C\{s_k\}$ implies that $F_\mu - F_\nu$ is identically zero on $\mathbb{R}$. Consequently, Proposition 4.2.3-a) ensures that $\mu = \nu$. □

Carleman's condition (4.1) is only sufficient for the $\mathbb{R}-$determinacy. Indeed, there exist $\mathbb{R}-$determinate measures whose moments do not fulfilll Carleman's condition (see [53] for examples).

As a consequence of Carleman's Theorem, we can derive a sufficient condition for the $(\mathbb{R}^+)-$determinacy.

Corollary 4.2.4.
Let $\mu \in \mathcal{M}^*(\mathbb{R}^+)$. If

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2k]{m_k^\mu}} = \infty, \tag{4.3}$$

then μ is $(\mathbb{R}^+)-$determinate realizing m.

Condition (4.3) is well-know as *Stieltjes' condition* since it is sufficient for the determinacy of the Stieltjes moment problem.

Before providing the proof of Corollary 4.2.4, recall that the image measure of a measure μ on $\mathcal{B}(\mathbb{R}^n)$ through a given Borel measurable map $\varphi : \mathbb{R}^n \to \mathbb{R}^d$ $(n, d \in \mathbb{N})$ is the measure $\varphi\#\mu$ on $\mathcal{B}(\mathbb{R}^d)$ defined by $\varphi\#\mu(B) := \mu(\varphi^{-1}(B))$ for all $B \in \mathcal{B}(\mathbb{R}^d)$. Moreover, for any $g : \mathbb{R}^d \to \mathbb{R}$ integrable w.r.t. $\varphi\#\mu$ we have that

$$\int_{\mathbb{R}^d} g(y) d(\varphi\#\mu)(y) = \int_{\mathbb{R}^n} (g \circ \varphi)(x) d\mu(x). \tag{4.4}$$

Proof.
Let $\mu_1, \mu_2 \in \mathcal{M}^*(\mathbb{R}^+)$ having the same moment sequence fulfilling Stieltjes' condition. For $j \in \{1, 2\}$ we define

$$d\nu_j(x) := \frac{1}{2}\left(f\#\mu_j + (-f)\#\mu_j)\right),$$

where $f : \mathbb{R}^+ \to \mathbb{R}$ is given by $f(x) := \sqrt{x}$. Then (4.4) implies that for any $k \in \mathbb{N}_0$ and any $j \in \{1, 2\}$ we have

$$\begin{aligned} m_{2k}^{\nu_j} &= \int_{\mathbb{R}} y^{2k} d\nu_j(y) = \frac{1}{2}\int_{\mathbb{R}} y^{2k} d(f\#\mu_j)(y) + \frac{1}{2}\int_{\mathbb{R}} y^{2k} d((-f)\#\mu_j)(y) \\ &= \frac{1}{2}\int_{\mathbb{R}^+} (\sqrt{x})^{2k} d\mu_j(x) + \frac{1}{2}\int_{\mathbb{R}^+} (-\sqrt{x})^{2k} d\mu_j(x) = \int_{\mathbb{R}^+} (\sqrt{x})^k d\mu_j(x) = m_k^{\mu_j}. \end{aligned}$$

and

$$\begin{aligned} m_{2k+1}^{\nu_j} &= \int_{\mathbb{R}} y^{2k+1} d\nu_j(y) = \frac{1}{2}\int_{\mathbb{R}} y^{2k+1} d(f\#\mu_j)(y) + \frac{1}{2}\int_{\mathbb{R}} y^{2k+1} d((-f)\#\mu_j)(y) \\ &= \frac{1}{2}\int_{\mathbb{R}^+} (\sqrt{x})^{2k+1} d\mu_j(x) + \frac{1}{2}\int_{\mathbb{R}^+} (-\sqrt{x})^{2k+1} d\mu_j(x) = 0. \end{aligned}$$

Then ν_1 and ν_2 have the same moments and

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2k]{m_{2k}^{\nu_j}}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[2k]{m_{k}^{\mu_j}}} = \infty.$$

Hence, Carleman's Theorem 4.2.1 ensures that $\nu_1 \equiv \nu_2$ on $\mathbb{R}$ and so $\mu_1 \equiv \mu_2$ on $\mathbb{R}^+$. □

Determinacy is also deeply connected to polynomial approximation. One result in this direction is the following, which will be particularly useful in the next section.

Lemma 4.2.5.
If $\mu \in \mathcal{M}^*(\mathbb{R})$ is $\mathbb{R}-$determinate, then $\mathbb{C}[x]$ is dense in $L^2(\mathbb{R}, \mu)$.

Proof. (see e.g. [50, Proposition 6.10]) □

4.3 Determinacy in higher dimensions

In this section we are going to prove a multivariate version of Carleman's Theorem 4.2.1, namely we give a proof of Theorem 3.3.9 which we restate here for the convenience of the reader.

Theorem 4.3.1. *Let $n \in \mathbb{N}$. If $\mu \in \mathcal{M}^*(\mathbb{R}^n)$ is s.t. its moment sequence $(m_\alpha^\mu)_{\alpha \in \mathbb{N}_0^n}$ fulfills*

$$\sum_{k=1}^{\infty} m^{\mu}_{(0,\ldots 0, \underbrace{2k}_{j\text{-}th}, 0,\ldots,0)}{}^{-\frac{1}{2k}} = \infty, \quad \forall j \in \{1, \ldots, n\}, \tag{4.5}$$

then μ is $(\mathbb{R}^n)-$determinate, i.e. the set

$$\mathcal{M}_\mu := \left\{ \nu \in \mathcal{M}^*(\mathbb{R}^n) : \int x^\alpha d\nu(x) = \int x^\alpha d\mu(x), \ \forall \alpha \in \mathbb{N}_0^n \right\}$$

is a singleton.

Note that the set $\mathcal{M}_\mu$ is convex and we have the following characterization of its extreme points[1].

[1] Recall that ν is an extreme point of $\mathcal{M}_\mu$ if the following implication holds: $(\nu = \lambda\eta_1 + (1 - \lambda\eta_2)$, for some $\lambda \in [0, 1], \eta_1, \eta_2 \in \mathcal{M}_\mu) \Rightarrow (\nu = \eta_1$ or $\nu = \eta_2)$.

Lemma 4.3.2. *Let $\mu, \nu \in \mathcal{M}^*(\mathbb{R}^n)$. Then ν is an extreme point of $\mathcal{M}_\mu$ if and only if $\mathcal{C}[X_1, \ldots, X_n]$ is dense in $L^1(\mathbb{R}^n, \nu)$.*

Proof. (see e.g. [50, Proposition 1.21]) □

To prove Theorem 4.3.1, we can proceed in the two following ways:

- We generalize the theory of quasi-analytic functions to the higher dimensions and prove an analogue of the Denjoy-Carleman theorem in the multivariate case. Using such results, we adapt the proof of Carleman's Theorem 4.2.1 to the higher dimensional case and provide a proof of Theorem 4.3.1 (see [26]).
- Using the connection between determinacy and polynomial approximation, we prove the so-called Petersen's theorem [39] about partial determinacy and so to reduce the $(\mathbb{R}^n)-$determinacy question to several $\mathbb{R}-$determinacy questions. Combining this result with Carleman's Theorem 4.2.1, we show that Theorem 4.3.1 holds (see [41]).

As we have already seen the power of the theory of quasi-analytic functions in the study of the determinacy question in the one-dimensional case, we are going now to use the second approach for the higher dimensional case. Therefore, let us first show Petersen's theorem.

Theorem 4.3.3 (Petersen's Theorem)**.**
Let $\mu \in \mathcal{M}^*(\mathbb{R}^n)$ and for each $j \in \{1, \ldots, n\}$ define $\pi_j(x) := x_j$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. If $\pi_1\#\mu, \ldots, \pi_n\#\mu$ are all $\mathbb{R}-$determinate, then μ is $(\mathbb{R}^n)-$determinate.

Proof.
Let $\nu \in \mathcal{M}_\mu$ and $j \in \{1, \ldots, n\}$. Then for any $k \in \mathbb{N}_0$ we have that

$$\begin{aligned}
\int_{\mathbb{R}} y^k d(\pi_j\#\nu)(y) &= \int_{\mathbb{R}^n} \pi_j(x)^k d\nu(x) \\
&= \int_{\mathbb{R}^n} x^{(0,\ldots,0,k,0,\ldots,0)} d\nu(x) \\
&= \int_{\mathbb{R}^n} x^{(0,\ldots,0,k,0,\ldots,0)} d\mu(x) \\
&= \int_{\mathbb{R}^n} \pi_j(x)^k d\mu(x) \\
&= \int_{\mathbb{R}} y^k d(\pi_j\#\mu)(y),
\end{aligned}$$

i.e. $(\pi_j \# \nu) \in \mathcal{M}_{\pi_j \# \mu}$. This implies that

$$\pi_j \# \nu = \pi_j \# \mu \tag{4.6}$$

as $\pi_j \# \mu$ is $\mathbb{R}-$determinate. Moreover, the determinacy of $\pi_j \# \mu$ implies that $\mathbb{C}[X_j]$ is dense in $L^2(\mathbb{R}, \mu)$ by Lemma 4.2.5 and so that

$$\forall \varepsilon > 0, \forall B_j \in \mathcal{B}(\mathbb{R}), \exists\, p_j \in \mathbb{C}[X_j] \text{ s.t. } \left\| \mathbb{1}_{B_j} - p_j \right\|_{L^2(\mathbb{R}, \pi_j \# \mu)} \leq \varepsilon. \tag{4.7}$$

Since

$$\begin{aligned}
\left\| \mathbb{1}_{B_j} - p_j \right\|_{L^2(\mathbb{R}, \pi_j \# \mu)} &\overset{(4.6)}{=} \left\| \mathbb{1}_{B_j} - p_j \right\|_{L^2(\mathbb{R}, \pi_j \# \nu)} \\
&= \left(\int_{\mathbb{R}} (\mathbb{1}_{B_j}(y) - p_j(y))^2 d(\pi_j \# \nu)(y) \right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{R}^n} (\mathbb{1}_{B_j}(\pi_j(x)) - p_j(\pi_j(x)))^2 d\nu(x) \right)^{\frac{1}{2}} \\
&= \left\| \mathbb{1}_{B_j} \circ \pi_j - p_j \circ \pi_j \right\|_{L^2(\mathbb{R}^n, \nu)},
\end{aligned}$$

we can rewrite (4.7) as

$$\forall \varepsilon > 0, \forall B_j \in \mathcal{B}(\mathbb{R}), \exists\, p_j \in \mathbb{C}[X_j] \text{ s.t. } \left\| \mathbb{1}_{B_j} \circ \pi_j - p_j \circ \pi_j \right\|_{L^2(\mathbb{R}^n, \nu)} \leq \varepsilon. \tag{4.8}$$

Now the function $(\mathbb{1}_{B_1} \circ \pi_1) \cdots (\mathbb{1}_{B_n} \circ \pi_n) - (p_1 \circ \pi_1) \cdots (p_n \circ \pi_n)$ on $\mathbb{R}^n$ can be rewritten as

$$\begin{aligned}
&(\mathbb{1}_{B_1} \circ \pi_1) \cdots (\mathbb{1}_{B_n} \circ \pi_n) - (p_1 \circ \pi_1) \cdots (p_n \circ \pi_n) = \\
&(\mathbb{1}_{B_1} \circ \pi_1 - p_1 \circ \pi_1)(\mathbb{1}_{B_2} \circ \pi_2) \cdots (\mathbb{1}_{B_n} \circ \pi_n) + \\
&\qquad + \quad (p_1 \circ \pi_1)(\mathbb{1}_{B_2} \circ \pi_2 - p_2 \circ \pi_2)(\mathbb{1}_{B_3} \circ \pi_3) \cdots (\mathbb{1}_{B_n} \circ \pi_n) + \\
&\qquad + \cdots + \quad (p_1 \circ \pi_1) \cdots (p_{n-1} \circ \pi_{n-1})(\mathbb{1}_{B_n} \circ \pi_n - p_n \circ \pi_n).
\end{aligned} \tag{4.9}$$

and so

$$\begin{aligned}
&\|(\mathbb{1}_{B_1} \circ \pi_1) \cdots (\mathbb{1}_{B_n} \circ \pi_n) - (p_1 \circ \pi_1) \cdots (p_n \circ \pi_n)\|_{L^1(\mathbb{R}^n, \nu)} \\
\overset{(4.9)}{\leq} \quad & \|(\mathbb{1}_{B_1} \circ \pi_1 - p_1 \circ \pi_1)(\mathbb{1}_{B_2} \circ \pi_2) \cdots (\mathbb{1}_{B_n} \circ \pi_n)\|_{L^1(\mathbb{R}^n, \nu)} + \cdots \\
\cdots \quad & + \|(p_1 \circ \pi_1) \cdots (p_{n-1} \circ \pi_{n-1})(\mathbb{1}_{B_n} \circ \pi_n - p_n \circ \pi_n)\|_{L^1(\mathbb{R}^n, \nu)} \\
\overset{\text{Hölder}}{\leq} \quad & \|\mathbb{1}_{B_1} \circ \pi_1 - p_1 \circ \pi_1\|_{L^2(\mathbb{R}^n, \nu)} \|(\mathbb{1}_{B_2} \circ \pi_2) \cdots (\mathbb{1}_{B_n} \circ \pi_n)\|_{L^2(\mathbb{R}^n, \nu)} + \cdots \\
\cdots \quad & + \|(p_1 \circ \pi_1) \cdots (p_{n-1} \circ \pi_{n-1})\|_{L^2(\mathbb{R}^n, \nu)} \|\mathbb{1}_{B_n} \circ \pi_n - p_n \circ \pi_n\|_{L^2(\mathbb{R}^n, \nu)} \\
\overset{(4.8)}{\leq} \quad & C\varepsilon,
\end{aligned}$$

where $C > 0$.

This shows that $\mathbb{C}[X_1, \ldots, X_n]$ is dense in the subset

$$\mathcal{S} := \{(\mathbb{1}_{B_1} \circ \pi_1) \cdots (\mathbb{1}_{B_n} \circ \pi_n) : B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}^n)\}$$

of $L^1(\mathbb{R}^n, \nu)$. Since $\operatorname{span}(\mathcal{S})$ is dense in $L^1(\mathbb{R}^n, \nu)$, we get that $\mathbb{C}[X_1, \ldots, X_n]$ is dense in $L^1(\mathbb{R}^n, \nu)$ and so by Lemma 4.3.2 we obtain that ν is an extreme point of $\mathcal{M}_\mu$.

Since ν was arbitrary in $\mathcal{M}_\mu$, we have showed that every point of $\mathcal{M}_\mu$ is extreme. In particular, $\eta := \frac{1}{2}(\mu + \nu) \in \mathcal{M}_\mu$ is extreme and so $\eta = \mu$ or $\eta = \nu$, which imply $\nu = \mu$. Hence, μ is $(\mathbb{R}^n)-$determinate. □

Proof. of Theorem 4.3.1
For any $j \in \{1, \ldots, n\}$ and for any $k \in \mathbb{N}$ we have that

$$\begin{aligned} m_{2k}^{\pi_j \# \mu} &= \int_{\mathbb{R}} y^{2k} d(\pi_j \# \mu)(y) = \int_{\mathbb{R}^n} (\pi_j(x))^{2k} d\mu(x) \\ &= \int_{\mathbb{R}^n} x^{(0,\ldots,0,2k,0,\ldots,0)} d\mu(x) = m^{\mu}_{(0,\ldots,0,2k,0,\ldots,0)}. \end{aligned}$$

Hence, the assumption that μ fulfils (4.5) gives that each $\pi_j \# \mu$ fulfils (4.1). Therefore, Carleman's Theorem 4.2.1 guarantees that each $\pi_j \# \mu$ is $\mathbb{R}-$determinate and so by Petersen's Theorem 4.3.3 we obtain that μ is $(\mathbb{R}^n)-$determinate. □

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