FORMS ON INNER PRODUCT SPACES

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Abstract. This note aims to give an introduction on forms on inner product spaces and their relation to linear operators. After briefly recalling some basic concepts from the theory of linear operators on inner product spaces, we focus on the space of forms on a real or complex finite-dimensional vector space V and show that it is isomorphic to the space of linear operators on V. We also describe the matrix representation of a form with respect to an ordered basis of the space on which it is defined, giving special attention to the case of forms on finite-dimensional complex inner product spaces and in particular to Hermitian forms.

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Introduction

In this note we are going to introduce the concept of forms on an inner product space and describe some of their main properties (c.f. [2, Section 9.2]). The notion of inner product is a basic notion in linear algebra which allows to rigorously introduce on any vector space intuitive geometrical notions such as the length of an element and the orthogonality between two of them. We will just recall this notion in Section 1 together with some basic examples (for more details on this structure see e.g. [1, Chapter 3], [2, Chapter 8], [3]). In Section 1 we will also recall the famous Riesz' representation theorem for finite-dimensional inner product spaces which describes a direct correspondence between inner products and linear functionals. This will be particularly useful in the following because we are going to analyze in details the relation between forms and linear operators on inner product spaces. Indeed, after introducing the general definition of a form in Section 2, we prove the main result of this note (see Theorem 2.4). This establishes an isomorphism between the space of all forms on a real or complex finite-dimensional vector space V and the space of linear operators on V. In Section 3 we define the matrix associated to a form on a finite-dimensional vector space w.r.t. an ordered basis of its domain and study some properties occurring in the complex case in presence of an inner product. In particular, we will see how the isomorphism mentioned above allows to transfer fundamental properties of representing matrices of linear operators to forms. This same direction is pursued in Section 4, where we focus on the class of Hermitian forms and derive a beautiful correspondence between Hermitian forms and self-adjoint linear operators. On the one hand, we derive from a basic property of Hermitian forms a characterization of self-adjoint linear operators on finite-dimensional complex inner product spaces. On the other hand, thanks to a fundamental property of linear self-adjoint operators, we give a quite direct proof of the so-called *principal axis theorem*.

1. Preliminaries

Throughout this note we denote by $\mathbb{K}$ the field of real numbers or the field of complex numbers and consider only vector spaces over $\mathbb{K}$.

Let us start by recalling the definition of inner product and making some examples of inner product spaces.

Definition 1.1 (Inner product)**.**
Let V be a vector space over $\mathbb{K}$. A function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$ is an *inner product* on V if for all $\alpha, \beta, \gamma \in V$ and for all $c \in \mathbb{K}$ the following properties hold:

(a) $\langle c\alpha + \beta, \gamma\rangle = c\langle\alpha, \gamma\rangle + \langle\beta, \gamma\rangle$ (linearity in the first variable)
(b) $\langle\beta, \alpha\rangle = \overline{\langle\alpha, \beta\rangle}$ (conjugate symmetry)
(c) $\langle\alpha, \alpha\rangle \geq 0$ and $\langle\alpha, \alpha\rangle = 0 \Leftrightarrow \alpha = 0$ (positive definiteness)

Remark 1.2. Note that conditions (a), (b) imply the conjugate linearity of the inner product in the second variable, i.e. for all $\alpha, \beta, \gamma \in V$

$$\langle\alpha, c\beta + \gamma\rangle = \overline{c}\langle\alpha, \beta\rangle + \langle\alpha, \gamma\rangle.$$

Definition 1.3 (Inner product space)**.**
A vector space V over $\mathbb{K}$ together with an inner product $\langle\cdot,\cdot\rangle$ is said to be an *inner product space* and it is usually denoted by $(V, \langle\cdot,\cdot\rangle)$.

A finite-dimensional inner product space over $\mathbb{K}$ is called *euclidean space* when $\mathbb{K} = \mathbb{R}$ and *unitary space* when $\mathbb{K} = \mathbb{C}$.

Examples 1.4.
Let $n \in \mathbb{N}$.

1. Consider the function $\langle \cdot, \cdot \rangle : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ defined by:
$$\langle \alpha, \beta \rangle := \sum_{i=1}^{n} x_i \overline{y_i}, \quad \forall\, \alpha = (x_1, \ldots, x_n)^t, \beta = (y_1, \ldots, y_n)^t \in \mathbb{K}^n, \tag{1.1}$$
where t denotes the transpose operator. It is easy to check that $(\mathbb{K}^n, \langle \cdot, \cdot \rangle)$ is an inner product space. The inner product defined in (1.1) is usually called *standard inner product*.
2. Let $\mathcal{M}(n \times n; \mathbb{K})$ be the space of all square matrices of order n with entries in $\mathbb{K}$ and for any $B = (B_{jk})_{j,k=1}^n$ we define the *conjugate traspose* B^* of B to be the matrix $B^* := (B^*_{jk})_{j,k=1}^n$ with $B^*_{jk} = \overline{B_{kj}}$. Consider the function $\langle \cdot, \cdot \rangle : \mathcal{M}(n \times n; \mathbb{K}) \times \mathcal{M}(n \times n; \mathbb{K}) \to \mathbb{K}$ defined by:
$$\langle A, B \rangle := tr(AB^*), \quad \forall A, B \in \mathcal{M}(n \times n; \mathbb{K}),$$
where tr denotes the trace operator on $\mathcal{M}(n \times n; \mathbb{K})$. Then for any $A = (A_{jk})_{j,k=1}^n, B = (B_{jk})_{j,k=1}^n \in \mathcal{M}(n \times n; \mathbb{K})$ we have
$$\langle A, B \rangle = tr(AB^*) = \sum_{j=1}^{n} (AB^*)_{jj} = \sum_{j=1}^{n} \sum_{k=1}^{n} A_{jk} B^*_{kj} = \sum_{j=1}^{n} \sum_{k=1}^{n} A_{jk} \overline{B_{jk}}.$$
It is now clear that the latter corresponds to the standard inner product on $\mathbb{K}^{n^2}$ through the well-known isomorphism between $\mathcal{M}(n \times n; \mathbb{K})$ and $\mathbb{K}^{n^2}$. Hence, $(\mathcal{M}(n \times n; \mathbb{K}), \langle \cdot, \cdot \rangle)$ is an inner product space. (Of course one can also directly check that $\langle \cdot, \cdot \rangle$ satisfies all the properties (a), (b), (c) of Definition 1.1 to show that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{M}(n \times n; \mathbb{K})$).

Exercise 1.5. Consider the space $\mathcal{C}([0, 1])$ of all continuous functions from $[0, 1]$ to $\mathbb{C}$ and the function $\langle \cdot, \cdot \rangle : \mathcal{C}([0, 1]) \times \mathcal{C}([0, 1]) \to \mathbb{C}$ defined by:
$$\langle f, g \rangle := \int_0^1 f(x) \overline{g(x)} dx, \quad \forall\, f, g \in \mathcal{C}([0, 1]). \tag{1.2}$$
Show that $\langle \cdot, \cdot \rangle$ is a complex inner product on $\mathcal{C}([0, 1])$. [1]

Before introducing the concept of form on a generic inner product space, let us recall an important result about linear functionals on finite-dimensional inner product spaces.

Theorem 1.6 (Riesz' representation theorem)**.**
Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product space over $\mathbb{K}$ and $T : V \to \mathbb{K}$ linear. Then there exists a unique $\beta \in V$ such that $T(\alpha) = \langle \alpha, \beta \rangle$ for all $\alpha \in V$.

Proof. (see [2, Section 8.3, Theorem 6] or [3, Skript 20, Satz 3]) ■

The Riesz' representation theorem guarantees that every linear functional on a finite-dimensional inner product space is the inner product w.r.t. a fixed vector of its domain. This result can be used to prove the existence of the *adjoint* T^* of a linear operator T on a finite-dimensional inner product space V over $\mathbb{K}$ (see [2, Section 8.3, Theorem 7]), i.e. $T^* : V \to V$ linear such that $\langle T\alpha, \beta \rangle = \langle \alpha, T^*\beta \rangle$,

$\forall\, \alpha, \beta \in V$. Recall also that a linear operator T on V is said to be *self-adjoint* if $T = T^*$.

As a last reminder, we recall the notion of matrix associated to a linear operator on a finite-dimensional vector space V over $\mathbb{K}$. Let us denote by $\mathcal{L}(V;V)$ the space of all linear operators from V to V.

Definition 1.7. Let $n \in \mathbb{N}$. Given an orderded basis $\mathcal{B} := \{\alpha_1, \ldots, \alpha_n\}$ of an $n-$dimensional vector space V over $\mathbb{K}$ and $T \in \mathcal{L}(V;V)$, the matrix $[T]_{\mathcal{B}}$ having as columns the vectors $T\alpha_j$ for $j = 1, \ldots, n$ is called the *matrix associated to T w.r.t. $\mathcal{B}$*. In symbols $[T]_{\mathcal{B}} := (A_{jk})_{j,k=1}^n$ s.t. $T\alpha_j = \sum_{k=1}^n A_{kj}\alpha_k$, $\forall\, j \in \{1, \ldots, n\}$.

2. Sesquilinear forms on inner product spaces

Definition 2.1. A (sesquilinear) form on a vector space V over $\mathbb{K}$ is a function $f : V \times V \to \mathbb{K}$ such that: for all $\alpha, \beta, \gamma \in V$ and for all $c \in \mathbb{K}$ the following hold:
(a) $f(c\alpha + \beta, \gamma) = cf(\alpha, \gamma) + f(\beta, \gamma)$
(b) $f(\alpha, c\beta + \gamma) = \bar{c}f(\alpha, \beta) + f(\alpha, \gamma)$.

Thus, a sesquilinear form f on a vector space V over $\mathbb{K}$ is a function on $V \times V$ which is linear as a function of the first argument but conjugate linear as a function of the second argument. Note that if $\mathbb{K} = \mathbb{R}$ then any sequilinear form f is linear as a function of each of its arguments, i.e. f is a bilinear form. However, in the case $\mathbb{K} = \mathbb{C}$, a sesquilinear form f is not bilinear unless $f \equiv 0$ on V. In the following, for notational convenience, we will omit the adjective *sesquilinear*.

Example 2.2. Any inner product on a vector space V over $\mathbb{K}$ is a form on V.

Exercise 2.3. For all $\alpha = (x_1, x_2)^t, \beta = (y_1, y_2)^t \in \mathbb{C}^2$, let
1. $f(\alpha, \beta) := 1$
2. $g(\alpha, \beta) := (x_1 - \overline{y_1})^2 + x_2\overline{y_2}$
3. $h(\alpha, \beta) := (x_1 + \overline{y_1})^2 - (x_1 - \overline{y_1})^2$.

Establish which ones of the above functions are forms on $\mathbb{C}^2$, motivating your answers. [2]

Let us denote by $\mathcal{F}(V, V; \mathbb{K})$ the space of all forms on V. This is a linear subspace of the vector space of all scalar valued functions on $V \times V$.

The following result summarizes the beautiful relation existing between forms and linear operators on finite-dimensional inner product spaces.

Theorem 2.4. *Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product space over $\mathbb{K}$.*
(a) If $f \in \mathcal{F}(V, V; \mathbb{K})$, then there exists a unique linear operator $T_f : V \to V$ such that $f(\alpha, \beta) = \langle T_f\alpha, \beta\rangle, \quad \forall\, \alpha, \beta \in V$.
(b) The map $\phi : \mathcal{F}(V, V; \mathbb{K}) \to \mathcal{L}(V;V)$ defined by $\phi(f) := T_f$ for all $f \in \mathcal{F}(V, V; \mathbb{K})$ is an isomorphism of vector spaces.

Proof.
(a) Existence: Let us fix a vector $\beta \in V$. Then

$$\begin{aligned} f_\beta :& V \to \mathbb{K} \\ & \alpha \mapsto f_\beta(\alpha) := f(\alpha, \beta) \end{aligned}$$

is a linear functional on V and so by Theorem 1.6 there exists a unique $\beta' \in V$ s.t. $f_\beta(\alpha) = \langle \alpha, \beta' \rangle$, $\forall\, \alpha \in V$.

Define $U : V \to V$ by $U(\beta) := \beta'$, $\forall \beta \in V$. Then we have

$$f(\alpha, \beta) = \langle \alpha, U(\beta)\rangle, \ \forall \alpha, \beta \in V. \tag{2.1}$$

Since $f \in \mathcal{F}(V, V; \mathbb{K})$, we know that for all $\alpha, \beta, \gamma \in V$ and for all $c \in \mathbb{K}$:

$$f(\alpha, c\beta + \gamma) = \overline{c} f(\alpha, \beta) + f(\alpha, \gamma)$$

which, by using (2.1), can be rewritten as:

$$\langle \alpha, U(c\beta + \gamma)\rangle = \overline{c}\langle \alpha, U(\beta)\rangle + \langle \alpha, U(\gamma)\rangle = \langle \alpha, cU(\beta) + U(\gamma)\rangle.$$

The latter implies that

$$U(c\beta + \gamma) = cU(\beta) + U(\gamma)\beta, \quad \forall\, \beta, \gamma \in V, \forall\; c \in \mathbb{K}$$

i.e. U is linear. Since V is finite-dimensional, then there exists the adjoint U^* of the linear operator U on V. Taking $T_f := U^*$, we have that for all $\alpha, \beta \in V$

$$f(\alpha, \beta) \overset{(2.1)}{=} \langle \alpha, U(\beta)\rangle = \langle U^*(\alpha), \beta\rangle = \langle T_f(\alpha), \beta\rangle,$$

which is the desired conclusion.
Uniqueness: Suppose there exists $T \in \mathcal{L}(V; V)$ s.t. $T \neq T_f$, $f(\alpha, \beta) = \langle T(\alpha), \beta\rangle$ $\forall \alpha, \beta \in V$. Then

$$0 = \langle T_f(\alpha), \beta\rangle - \langle T(\alpha), \beta\rangle = \langle T_f(\alpha) - T(\alpha), \beta\rangle, \quad \forall\, \alpha, \beta \in V$$

and so $T_f\alpha = T\alpha$, $\forall \alpha \in V$, which is a contradiction.

(b) To show that ϕ is an isomorphism between $F(V, V; \mathbb{K})$ and $\mathcal{L}(V; V)$ we need to prove that ϕ is a linear bijective map between these two vector spaces. Part (a) already shows that ϕ is bijective so it remains to verify the linearity of ϕ. Let $f, g \in F(V, V; \mathbb{K})$ and $c \in \mathbb{K}$. Then for any $\alpha, \beta \in V$ we have

$$\begin{aligned}\langle T_{cf+g}\alpha, \beta\rangle &\overset{(a)}{=} (cf + g)(\alpha, \beta) = cf(\alpha, \beta) + g(\alpha, \beta)\\ &\overset{(a)}{=} c\langle T_f\alpha, \beta\rangle + \langle T_g\alpha, \beta\rangle = \langle (cT_f + T_g)\alpha, \beta\rangle\end{aligned}$$

which implies that $T_{cf+g} = cT_f + T_g$, i.e. $\phi(cf + g) = c\phi(f) + \phi(g)$.

■

3. Matrix representation of a form

In this section we focus on forms on finite-dimensional inner product spaces and study their matrix representation. Let us start with a general definition.

Definition 3.1. Let $n \in \mathbb{N}$. Given an ordered basis $\mathcal{B} := \{\alpha_1, \ldots, \alpha_n\}$ of an $n-$dimensional vector space V over $\mathbb{K}$ and $f \in F(V, V; \mathbb{K})$, the matrix $[f]_{\mathcal{B}} := (A_{jk})_{j,k=1}^n$ defined by $A_{jk} := f(\alpha_k, \alpha_j)$ for all $j, k = 1, \ldots, n$ is called the *matrix associated to f w.r.t. $\mathcal{B}$.*

Proposition 3.2. *Let V be a finite-dimensional inner product space over $\mathbb{K}$ and $f \in F(V, V; \mathbb{K})$. If $\mathcal{B}$ is an orthonormal basis of V, then $[f]_{\mathcal{B}} \equiv [T_f]_{\mathcal{B}}$ (here T_f is the operator given by Theorem 2.4 and $[T_f]_{\mathcal{B}}$ is defined according to Definition 1.7).*

Proof. Let $\mathcal{B} := \{\alpha_1, \ldots, \alpha_n\}$ be an orthonormal basis of V. Let $[f]_{\mathcal{B}} := (A_{jk})_{j,k=1}^n$ and $[T_f]_{\mathcal{B}} := (M_{jk})_{j,k=1}^n$. Then for any $j, k \in \{1, \ldots, n\}$ we have:

$$A_{jk} \overset{\text{Def 3.1}}{=} f(\alpha_k, \alpha_j) \overset{\text{Thm 2.4}}{=} \langle T_f\alpha_k, \alpha_j\rangle \overset{\text{Def 1.7}}{=} \langle \sum_{l=1}^n M_{lk}\alpha_l, \alpha_j\rangle = \sum_{l=1}^n M_{lk}\langle \alpha_l, \alpha_j\rangle = M_{jk}.$$

■

Note that Proposition 3.2 does not hold for a general basis as it is showed by the following example.

Example 3.3. Consider the standard inner product on $\mathbb{R}^2$ and $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := x_1y_2 + x_2y_1$ for all $x = (x_1, x_2)^t, y = (y_1, y_2)^t \in \mathbb{R}^2$. Let $\mathcal{B} := \left\{ b_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, b_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $\mathcal{B}' := \left\{ b_1' := \begin{pmatrix} 1 \\ 2 \end{pmatrix}, b_2' := \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$. Then $\mathcal{B}$ and $\mathcal{B}'$ are both bases of $\mathbb{R}^2$ but only $\mathcal{B}$ is also orthonormal. By using Definition 3.1, we obtain $[f]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $[f]_{\mathcal{B}'} = \begin{pmatrix} 4 & 10 \\ 10 & 24 \end{pmatrix}$. By Theorem 2.4-(a) applied to f we have that $T_f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T_f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$ and, by Definition 1.7, we can compute: $[T_f]_{\mathcal{B}} = \Big(T_f(b_1) \;\; T_f(b_2) \Big) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $[T_f]_{\mathcal{B}'} = \Big(T_f(b_1') \;\; T_f(b_2') \Big) = \begin{pmatrix} \frac{1}{2} & -\frac{7}{2} \\ \frac{1}{2} & \frac{5}{2} \end{pmatrix}$. Hence, $[T_f]_{\mathcal{B}} \equiv [f]_{\mathcal{B}}$ but $[T_f]_{\mathcal{B}'} \not\equiv [f]_{\mathcal{B}'}$.

Let $n \in \mathbb{N}$. If $\mathcal{B} := \{\alpha_1, \ldots, \alpha_n\}$ is an ordered basis of an $n-$dimensional vector space V over $\mathbb{K}$, $f \in F(V, V; \mathbb{K})$ and $[f]_{\mathcal{B}} = (A_{jk})_{j,k=1}^n$, then for any $x, y \in V$ we have $x = \sum_{s=1}^n x_s\alpha_s$, $y = \sum_{r=1}^n y_r\alpha_r$ and

$$f(x, y) = \sum_{r,s=1}^{n} \overline{y_r} f(\alpha_s, \alpha_r) x_s = \begin{pmatrix} \overline{y_1} & \ldots & \overline{y_n} \end{pmatrix} \begin{pmatrix} A_{11} & \ldots & A_{1n} \\ \vdots & \ldots & \vdots \\ A_{n1} & \ldots & A_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

i.e.

$$f(x, y) = Y^*[f]_{\mathcal{B}}X$$

where $X := [x]_{\mathcal{B}}$ and $Y := [y]_{\mathcal{B}}$ are the coordinate matrices of x and y w.r.t. $\mathcal{B}$ and Y^* is the conjugate transpose of Y (see Example 1.4-2).

For the remainder of this note, we want to concentrate on the complex finite-dimensional inner product spaces. In particular, in the last part of this section we want to show how the isomorphism described in Theorem 2.4 allows to transfer a very important property of representing matrices of linear operators to forms.

For convenience let us rewrite here the statement of the so-called *orthormal triangularizability* theorem for linear operators on complex finite-dimensional inner product spaces (see [2, Section 8.5, Theorem 21] or [3, Skript 23, Satz I]).

Theorem 3.4. *Let V be a finite-dimensional inner product space over $\mathbb{C}$ and let $T \in \mathcal{L}(V; V)$. Then there exists an orthonormal basis $\mathcal{B}$ for V s.t. $[T]_{\mathcal{B}}$ is an upper triangular matrix.*

Then we can easily derive the analogous result for forms:

Theorem 3.5. *Let V be a finite-dimensional inner product space over $\mathbb{C}$ and let $f \in F(V, V; \mathbb{C})$. Then there exists an orthonormal basis $\mathcal{B}$ for V s.t. $[f]_{\mathcal{B}}$ is an upper triangular matrix.*

Proof. By Theorem 2.4, there exists a unique linear operator T_f such that $f(\alpha, \beta) = \langle T_f\alpha, \beta\rangle$ for all $\alpha, \beta \in V$. Therefore, by applying Theorem 3.4 to T_f, we get that there exists an orthonormal basis $\mathcal{B}$ for V s.t. $[T_f]_{\mathcal{B}}$ is an upper triangular matrix. But Proposition 3.2 guarantees that $[f]_{\mathcal{B}} \equiv [T_f]_{\mathcal{B}}$, which completes the proof. ∎

4. Hermitian forms

In this section we continue to study forms on complex finite-dimensional inner product spaces and in particular we focus on a special class of those, namely the *Hermitian forms*.

Definition 4.1 (Hermitian form)**.** A form f on a complex vector space V is called *Hermitian* if $f(\alpha, \beta) = \overline{f(\beta, \alpha)}$ for all $\alpha, \beta \in V$.

The following is a well-known characterization of Hermitian forms.

Theorem 4.2. *Let V be a complex vector space and $f \in F(V, V; \mathbb{C})$. Then f is Hermitian if and only if $f(\alpha, \alpha) \in \mathbb{R}$ for all $\alpha \in V$.*

Proof. Suppose that f be an Hermitian form on V. Then by definition $f(\alpha, \alpha) = \overline{f(\alpha, \alpha)}$ and so $f(\alpha, \alpha) \in \mathbb{R}$.

Conversely, assume that $f(\alpha, \alpha) \in \mathbb{R}$ for all $\alpha \in V$. Let $\alpha, \beta \in V$. Then:

$$\underbrace{f(\alpha+\beta, \alpha+\beta)}_{\in\mathbb{R}} = \underbrace{f(\alpha, \alpha)}_{\in\mathbb{R}} + f(\alpha, \beta) + f(\beta, \alpha) + \underbrace{f(\beta, \beta)}_{\in\mathbb{R}}$$

and so $f(\alpha, \beta) + f(\beta, \alpha) \in \mathbb{R}$ i.e.

$$f(\alpha, \beta) + f(\beta, \alpha) = \overline{f(\alpha, \beta)} + \overline{f(\beta, \alpha)}. \tag{4.1}$$

We also have that

$$\underbrace{f(\alpha+i\beta, \alpha+i\beta)}_{\in\mathbb{R}} = \underbrace{f(\alpha, \alpha)}_{\in\mathbb{R}} - if(\alpha, \beta) + if(\beta, \alpha) + \underbrace{f(\beta, \beta)}_{\in\mathbb{R}}$$

and so $if(\beta, \alpha) - if(\alpha, \beta) \in \mathbb{R}$ i.e.

$$-if(\alpha, \beta) + if(\beta, \alpha) = i\overline{f(\alpha, \beta)} - i\overline{f(\beta, \alpha)}. \tag{4.2}$$

Multiplying (4.2) by i and summing it side by side to (4.1) we get $f(\alpha, \beta) = \overline{f(\beta, \alpha)}$, i.e. f is Hermitian. ■

Thanks to Theorem 2.4, we get an interesting connection between Hermitian forms and self-adjoint operators on finite-dimensional inner product spaces over $\mathbb{C}$.

Proposition 4.3. *Let $(V, \langle\cdot,\cdot\rangle)$ be a complex finite-dimensional inner product space and $f \in F(V, V; \mathbb{C})$. Then f is Hermitian if and only if T_f is self-adjoint (here T_f is the one given by Theorem 2.4).*

Proof. By Theorem 2.4, we know that $f(\alpha, \beta) = \langle T_f\alpha, \beta\rangle$ for all $\alpha, \beta \in V$ and so

$$\overline{f(\beta, \alpha)} = \overline{\langle T_f\beta, \alpha\rangle} = \langle \alpha, T_f\beta\rangle = \langle T_f^*\alpha, \beta\rangle.$$

Hence, we have that f is Hermitian if and only if $T_f = T_f^*$. ■

Combining the previous two results, we directly obtain the following characterization of self-adjoint operators on a complex finite-dimensional inner product space.

Corollary 4.4. *Let $(V, \langle\cdot,\cdot\rangle)$ be a complex finite-dimensional inner product space and $T \in \mathcal{L}(V; V)$. Then T is self-adjoint if and only if $\langle T\alpha, \alpha\rangle \in \mathbb{R}$, $\forall \alpha \in V$.*

Example 4.5. The form f defined in Example 1.4-2 for $\mathbb{K} = \mathbb{C}$ is Hermitian. Indeed, $\forall\, A \in \mathcal{M}(n \times n, \mathbb{C})$, $f(A, A) = \sum_{j,k=1}^n A_{jk}^* A_{kj} = \sum_{j,k=1}^n \overline{A_{kj}} A_{kj} \in \mathbb{R}$ and so f is Hermitian by Theorem 4.2.

In the end, let us point out some properties of the matrix associated to an Hermitian form on a finite-dimensional vector space. First of all, *if $\mathcal{B}$ is an ordered basis of a finite-dimensional complex vector space V and $f \in \mathcal{F}(V, V; \mathbb{C})$ then f is Hermitian if and only if $A := [f]_{\mathcal{B}}$ is Hermitian.* Indeed, we have seen in the previous section that for any $x, y \in V$ we have $f(x, y) = Y^* AX$ where $X = [x]_{\mathcal{B}}$ and $Y = [y]_{\mathcal{B}}$ and so $\overline{f(y,x)} = \overline{X^*AY} = X^t\overline{AY} = (X^t\overline{AY})^t = \overline{Y}^t\overline{A}^tX = Y^*A^*X$. Hence, f is Hermitian if and only if $A = A^*$, i.e. $[f]_{\mathcal{B}}$ is Hermitian.

A nice property of self-adjoint operators which reflects in the theory of representing matrix of Hermitian forms is the following one.

Theorem 4.6. *Let V be a finite-dimensional complex vector space and $T \in \mathcal{L}(V; V)$ self-adjoint. Then there exists an orthonormal basis $\mathcal{B}$ for V consisting of eigenvectors for T.*

Proof. (see [2, Sec 8.5, Theorem 18] or [3, Skript 23, Kor 2]) ■

This provides the following result for Hermitian forms which is often known as *Principle Axis Theorem.*

Theorem 4.7. *Let f be an Hermitian form on a finite-dimensional complex inner product space V. Then there exists an orthonormal basis $\mathcal{B}$ for V such that $[f]_{\mathcal{B}}$ is a diagonal matrix with real entries.*

Proof. Let $n \in \mathbb{N}$ be the dimension of V and $\langle \cdot, \cdot \rangle$ the inner product on V. By Theorem 2.4, there exists a unique linear operator T_f on V such that $f(\alpha, \beta) = \langle T_f\alpha, \beta\rangle$ for all $\alpha, \beta \in V$. Since f is Hermitian, by Proposition 4.3, we have that T_f is self-adjoint. Hence, by Theorem 4.6 there exists an orthonormal basis $\mathcal{B} := \{\alpha_1, \dots, \alpha_n\}$ for V consisting of eigenvectors for T_f, i.e. $T_f\alpha_j = c_j\alpha_j$ for $j = 1, \dots, n$ with $c_j \in \mathbb{C}$. Now for any $j, k \in \{1, \dots, n\}$ we have

$$f(\alpha_k, \alpha_j) = \langle T_f\alpha_k, \alpha_j\rangle = \langle c_k\alpha_k, \alpha_j\rangle = c_k\delta_{kj},$$

where δ_{jk} is the Korenecker delta. Hence, $[f]_{\mathcal{B}} = diag(c_1, \dots, c_n)$ and for any $j \in \{1, \dots, n\}$ we get $c_j = f(\alpha_j, \alpha_j)$, which is a real number by Theorem 4.2 as f is Hermitian. ■

Notes

1. **Solution to Ex 1.5**: For any $f, g, h \in \mathcal{C}([0, 1])$ and $c \in \mathbb{C}$ we have:
   (a) $\langle cf + g, h\rangle = \int_0^1 (cf(x) + g(x))\overline{h(x)}dx = c\int_0^1 f(x)\overline{h(x)}dt + \int_0^1 g(x)\overline{h(x)}dx = c\langle f, h\rangle + \langle g, h\rangle$
   (b) $\langle g, f\rangle = \int_0^1 g(x)\overline{f(x)}dx = \int_0^1 \overline{f(x)\overline{g(x)}}dx = \overline{\int_0^1 f(x)\overline{g(x)}dx} = \overline{\langle f, g\rangle}$
   (c) $\langle f, f\rangle = \int_0^1 f(x)\overline{f(x)}dx \geq 0$ since $f(x)\overline{f(x)} \geq 0$ for all $x \in [0, 1]$ and $\langle f, f\rangle = \int_0^1 f(x)\overline{f(x)}dx = 0$ iff $f \equiv 0$ on $[0, 1]$.

   Hence, $\langle \cdot, \cdot \rangle$ defined in (1.2) is a complex inner product on $\mathcal{C}([0, 1])$.
2. **Solution to Ex 2.3**: h is the only sesequilinear form on $\mathbb{C}^2$ among the given ones. In fact, it is easy to verify that f and g are both not linear in the first argument, while one can rewrite h as $h(\alpha, \beta) = 4x_1\overline{y_1}$ for all $\alpha = (x_1, x_2)^t, \beta = (y_1, y_2)^t \in \mathbb{C}^2$ which clearly fulfills all the properties in Definition 2.1.

References

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