Spectral Resolution and Functions of Diagonalisable Normal Operators

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Abstract

This note shall present an application of the spectral theorem to diagonalisable normal operators on finite-dimensional inner product spaces. After recalling the spectral resolution of such an operator T, we will show how a function f on its spectrum can be expanded to a new diagonalisable normal operator $f(T)$. We will describe some relations between $f(T)$ and T regarding their spectra and matrix representations and illustrate these by means of an example.

Contents

1 Introduction

The Spectral Theorem provides a canonical decomposition of diagonalisable normal operators into a linear combination of projections onto its eigenspaces. We will exploit this application of the Spectral Theorem to diagonalisable normal operators mainly following [1, Section 9.5]. In Section 2 we will briefly recall some preliminary definitions. Section 3 contains the Spectral Theorem both for self-adjoint and normal operators and further related statements. In Section 4 we will focus on diagonalisable normal operators. Given a scalar-valued function f on the spectrum of a diagonalisable normal operator T, we will explain how this can be expanded to an operator $f(T)$. We will prove that $f(T)$ is also a diagonalisable normal operator and some matrix representation results. Finally, we will illustrate these results by an example in $\mathbb{C}^2$.

2 Preliminaries

In this section we recall some basic terms and results which we need for the study of diagonalisable normal operators. These can be found in [1, Section 8].

Definition 2.1. Let V be a vector space over K, where $K = \mathbb{R}$ or $K = \mathbb{C}$. An *inner product* $(.\mid.)$ is a function from $V \times V$ to K satisfying for all $u, v, w \in V$ and $\lambda \in K$:

1. $(u + v \mid w) = (u \mid w) + (v \mid w)$,
2. $(\lambda v \mid w) = \lambda(v \mid w)$,
3. $(v \mid w) = \overline{(w \mid v)}$, where the bar denotes complex conjugation,
4. $(v \mid v) > 0$ if $v \neq 0$.

If $K = \mathbb{R}$, these properties describe a *positive definite symmetric bilinear form*; if $K = \mathbb{C}$, they describe a *positive definite hermitian sesquilinear form.*

The vector space V together with an inner product is called an *inner product space.*

We denote by $\mathcal{L}(V, V, K)$ the set of all linear operators on V, i. e. all endomorphisms on V.

Definition 2.2. Let V be an inner product space over $K = \mathbb{R}$ or $K = \mathbb{C}$. Let $T \in \mathcal{L}(V, V, K)$. Suppose that there exists $T^* \in \mathcal{L}(V, V, K)$ such that for all $v, w \in V$,

$$(Tv \mid w) = (v \mid T^*w).$$

Then T^* is called the *adjoint* of T.

Note that on finite-dimensional inner product spaces the adjoint of a linear operator always exists and is unique (see [1, Chapter 8, Theorem 7]).

Definition 2.3. Let V be a finite-dimensional inner product space over $K = \mathbb{R}$ or $K = \mathbb{C}$. Let $T \in \mathcal{L}(V, V, K)$. Then T is called *self-adjoint* if $T = T^*$, and T is called *normal* if it commutes with its adjoint, i. e. $T^*T = TT^*$.

Throughout this script, if not further specified, V is a finite-dimensional inner product space over either $K = \mathbb{R}$ or $K = \mathbb{C}$.

3 Spectral Resolution

In order to introduce the notion of the spectral resolution of a linear operator with certain properties, we need to state the Spectral Theorem. This is a consequence of the following two theorems on self-adjoint and normal operators.

Theorem 3.1. *Let $T \in \mathcal{L}(V, V, K)$ be self-adjoint. Then there exists an orthonormal basis for V which consists of eigenvectors of T.*

Proof. See [1, Chap. 8, Theorem 18]. □

Theorem 3.2. *Let $T \in \mathcal{L}(V, V, \mathbb{C})$ be normal. Then there exists an orthonormal basis for V which consists of eigenvectors of T.*

Proof. See [1, Chap. 8, Theorem 22] or [2, § 22, Korollar 2]. □

Recall that for an operator T on V with eigenvalue λ the eigenspace of T corresponding to λ is given by $\ker(T - \lambda I)$. Moreover, recall that for some orthogonal decomposition $V = W \oplus W^{\perp}$ the orthogonal projection $P : V \to W$ of V onto W is given by $P(v) = w$ where $v = w + w'$ with $w \in W$, $w' \in W^{\perp}$.

Theorem 3.3 (Spectral Theorem)**.** *Let $T \in \mathcal{L}(V, V, K)$ such that either $K = \mathbb{R}$ and T is self-adjoint or $K = \mathbb{C}$ and T is normal. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T. Let W_j be the eigenspace corresponding to λ_j. Let P_j be the orthogonal projection of V onto W_j. Then:*

1. *W_i is orthogonal to W_j if $i \neq j$,*
2. *$V = W_1 \oplus \ldots \oplus W_k$,*
3. *$T = \lambda_1 P_1 + \ldots + \lambda_k P_k$.*

Proof. See [1, Section 9.5, Theorem 9]. □

Definition 3.4. Let $T \in \mathcal{L}(V, V, K)$ satisfying the same conditions as in Theorem 3.3. The decomposition of T given in Theorem 3.3 − 3. is called the *spectral resolution* of T.

4 Functions of Diagonalisable Normal Operators

In order to establish the existence of a spectral resolution of a given $T \in \mathcal{L}(V, V, K)$, a sufficient condition is the existence of an orthonormal basis of V consisting of eigenvectors of T. If T is diagonalisable and normal (not depending on whether $K = \mathbb{R}$ or $K = \mathbb{C}$), then such an orthonormal basis exists. This is a direct consequence of the version of the Spectral Theorem stated in [2, §21]. Note that this also means that the conclusions of Theorem 3.2 and Theorem 3.3 hold in the case that $K = \mathbb{R}$ and T is diagonalisable and normal. We will henceforth focus on diagonalisable normal operators on finite-dimensional inner product spaces.

Recall that the spectrum of an operator T is defined as the set of its eigenvalues. We will denote the spectrum of T by $S(T)$.

Definition 4.1. Let $T \in \mathcal{L}(V, V, K)$ be normal and diagonalisable. Let

$$T = \lambda_1 P_1 + \ldots + \lambda_k P_k$$

be its spectral resolution. Let $f : S(T) \to K$ be a scalar-valued function on the spectrum of T. The linear operator $f(T) \in \mathcal{L}(V, V, K)$ is defined by

$$f(T) = f(\lambda_1) P_1 + \ldots + f(\lambda_k) P_k. \tag{1}$$

Definition 4.2. Let V and V' be inner product spaces over the same field K. A linear transformation $U \in \mathcal{L}(V, V', K)$ is called a *unitary transformation* if it is an isomorphism and it preserves the inner product, i. e. for all $v, w \in V$,

$$(v \mid w) = (Uv \mid Uw).$$

Theorem 4.3. *Let $T \in \mathcal{L}(V, V, K)$ be normal and diagonalisable, let $f : S(T) \to K$ and $f(T)$ be defined as in* (1)*. Then:*

1. *$f(T)$ is normal and diagonalisable;*
2. *$S(f(T)) = f(S(T))$;*
3. *if $U \in \mathcal{L}(V, V', K)$ is some unitary transformation and $T' = UTU^{-1}$, then $S(T) = S(T')$ and*
$$f(T') = Uf(T)U^{-1}.$$

Proof. 1. Let $v, w \in V$. Note that any orthogonal projection P_j is self-adjoint, as

$$(P_j v \mid w) = (P_j v \mid P_j w) = (v \mid P_j w).$$

Hence,

$$(f(T)v \mid w) = \sum_{j=1}^{k} f(\lambda_j)(P_j v \mid w) = \left(v \;\middle|\; \sum_{j=1}^{k} \overline{f(\lambda_j)} P_j w \right).$$

Hence, $f(T)^* = \sum_{j=1}^{k} \overline{f(\lambda_j)} P_j$. Note that

$$P_i P_j = \begin{cases} 0 & \text{if } i \neq j \\ P_i & \text{if } i = j. \end{cases}$$

Hence, $f(T)$ and $f(T)^*$ commute, i. e. $f(T)$ is normal.

If v is an eigenvector of T with eigenvalue λ, then $f(T)v = f(\lambda)v$. Hence, the basis for V consisting of eigenvectors of T already consists of eigenvectors of $f(T)$. This means that $f(T)$ is diagonalisable.

2. From the last step in the proof of 1., we obtain $f(S(T)) \subseteq S(f(T))$. Conversely, let $v \neq 0$ be an eigenvector of $f(T)$ with eigenvalue μ, i. e.

$$f(T)v = \mu v.$$

Then

$$f(T)v = \sum_{j=1}^{k} \mu P_j v$$

and

$$f(T)v = f(T) \sum_{j=1}^{k} P_j v = \sum_{j=1}^{k} f(\lambda_j) P_j v.$$

Hence,

$$0 = \left\| \sum_{j=1}^{k} (f(\lambda_j) - \mu) P_j v \right\|^2 = \sum_{j=1}^{k} |f(\lambda_j) - \mu|^2 \left\| P_j v \right\|^2.$$

Since $v \neq 0$, there must be some index i such that $P_i v \neq 0$ and thus $f(\lambda_i) = \mu$. Hence, $S(f(T)) \subseteq f(S(T))$.

3. Since $T' = UTU^{-1}$, the equation $Tv = \lambda v$ holds if and only if $T'Uv = \lambda Uv$. Hence, $S(T') = S(T)$ and U maps each eigenspace of T to the corresponding eigenspace of T'. Let $T' = \sum_{j=1}^{k} \lambda_j P'_j$ be the spectral resolution of T'. Then $P'_j = UP_jU^{-1}$. Hence,

$$f(T') = \sum_{j=1}^{k} f(\lambda_j)UP_jU^{-1} = U\left(\sum_{j=1}^{k} f(\lambda_j)P_j\right)U^{-1} = Uf(T)U^{-1}.$$

□

Remark 4.4. From the previous theorem we can also construct the spectral resolution of $f(T)$. Suppose that $f(S(T)) = \{\mu_1, \dots, \mu_\ell\}$. Note that $\ell \leq k$. For each $1 \leq m \leq \ell$ let J_m be the set of indices j such that $f(\lambda_j) = \mu_m$, and set

$$Q_m := \sum_{i \in J_m} P_i.$$

Then the spectral resolution of $f(T)$ is given by

$$f(T) = \sum_{m=1}^{\ell} \mu_m Q_m.$$

In the following corollary we slightly change the notation, as we are going to talk about not necessarily distinct eigenvalues of T. We denote these by μ_i.

Corollary 4.5. *Let $T \in \mathcal{L}(V, V, K)$ be normal and diagonalisable and $f : S(T) \to K$. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for V such that T is represented in $\mathcal{B}$ by a diagonal matrix D with diagonal entries μ_i, where μ_i is the eigenvalue of T corresponding to v_j. Then $f(T)$ is represented in $\mathcal{B}$ by the diagonal matrix $f(D)$ with entries $f(\mu_i)$. If $\mathcal{B}' = \{w_1, \dots, w_n\}$ is another basis for V and R is the base change matrix from $\mathcal{B}$ to $\mathcal{B}'$, i. e. $w_j = \sum_{i=1}^{n} R_{ij}v_i$, then $R^{-1}f(D)R$ is the matrix of $f(T)$ represented in $\mathcal{B}'$.*

Proof. The first part, that $f(T)$ is represented in $\mathcal{B}$ by the diagonal matrix $f(D)$ with entries $f(\mu_j)$, directly follows from Theorem 4.3.

For each index i denote by j_i the unique index such that $v_i \in P_{j_i}(V)$ and $\mu_i = \lambda_{j_i}$. Then

$$\begin{aligned}
f(T)w_{j_i} &= \sum_{i=1}^{n} P_{ij_i} f(T)v_i \\
&= \sum_{i=1}^{n} \mu_i P_{ij_i} v_i \\
&= \sum_{i=1}^{n} (DP)_{ij_i} v_i \\
&= \sum_{i=1}^{n} (DP)_{ij_i} \sum_{m=1}^{n} P^{-1}_{mi} w_m \\
&= \sum_{m=1}^{n} (P^{-1}DP)_{mj_i} w_m.
\end{aligned}$$

□

Example 4.6. Let $\mathbb{C}^2$ be equipped with its standard inner product. We consider the operator

$$T : \mathbb{C}^2 \to \mathbb{C}^2, \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} -\mathrm{i}b \\ \mathrm{i}a \end{pmatrix}.$$

Its representation in the canonical basis $\mathcal{E}$ is

$$[T]_{\mathcal{E}} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

We immediately see that T is self-adjoint and thus normal. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}, \begin{pmatrix} -\mathrm{i} \\ 1 \end{pmatrix} \right\}.$$

With respect to $\mathcal{B}$, the linear operator T is represented by the diagonal matrix

$$[T]_{\mathcal{B}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The projections onto the eigenspaces are given by

$$P_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{2}\mathrm{i} \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}, P_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix};$$

$$P_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\mathrm{i} \begin{pmatrix} -\mathrm{i} \\ 1 \end{pmatrix}, P_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\mathrm{i} \\ 1 \end{pmatrix}.$$

We obtain the spectral resolution $T = -P_1 + P_2$.

Now let $f : \{-1, 1\} \to \mathbb{C}$ be defined by $f(-1) = 1$ and $f(1) = \mathrm{i}$. Then $f(T) = P_1 + \mathrm{i}P_2$. This linear operator is represented with respect to $\mathcal{E}$ by

$$[f(T)]_{\mathcal{E}} = \frac{1}{2} \begin{pmatrix} 1+\mathrm{i} & 1+\mathrm{i} \\ -1-\mathrm{i} & 1+\mathrm{i} \end{pmatrix}.$$

One can by direct calculation verify that $f(T)$ is normal. Note that it is not self-adjoint! Finally, we obtain the diagonalisation of $f(T)$ by respresenting it with respect to $\mathcal{B}$:

$$[f(T)]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}.$$

References

[1] K. Hoffmann and R. Kunze, *Linear algebra*, second edition, Prentice-Hall, Englewood Cliffs, N. J., 1971.

[2] S. Kuhlmann, *Lineare Algebra II*, Gesamtskript zur Vorlesung, Universität Konstanz, Sommersemester 2016.