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REAL ALGEBRAIC GEOMETRY–WS 2014/15

Christmas Assignment

You do not need to hand in solutions for this assignment, but please try to solve as many questions as you can. ***This assignment is a really important training in preparation to your final exam!*** *The solutions of this assignment will be distributed on the 8th of January during the lecture break, so that you can check your answers and prepare yourself for discussing them on the 15th of January in the exercise group. The solutions will also be published on the webpage of the course:*
www.math.uni-konstanz.de/∼infusino/RAGWS14-15/RAGWS14-15-Exercises.html

1) Let $(K, \leq)$ be an ordered field.

a) Show that the system $\mathcal{I}$ of all intervals

$$]a, b[:= \{x \in K \mid a < x < b\}, \quad a, b \in K$$

is a cover of K and is closed under finite intersection.
Therefore, $\mathcal{I}$ is the base of a topology on K, which is called the *interval topology on* K. From now on, we consider K endowed with such a topology.

b) In Ex 3 a) of Sheet 7, we showed in particular that the field operations as mappings from $K \times K$ to K are continuous w.r.t. the product topology on $K \times K$ given by the interval topology on K. Prove that also the multiplicative inversion as mapping from $K^{\times} := K \setminus \{0\}$ to K is continuous w.r.t. the induced topology on $K^{\times}$.

c) Let $n \in \mathbb{N}$. Assume that K is real closed, then we can define the *euclidean norm* on K^n:

$$\begin{array}{cccc} \|\ \| : & K^n & \to & K \\ & (x_1, \ldots, x_n) & \to & \sqrt{x_1^2 + \cdots + x_n^2}. \end{array}$$

Show that the *euclidean topology on* K^n, i.e. the topology induced by the euclidean norm, coincides with the product topology on K^n given by the interval topology on K.

2) Let us introduce the following definition:

Definition 1.
Let $G := (G, \cdot)$ be a group. G is said to be an ***ordered group*** *if there exists a total ordering $\leq$ of the set G such that for any $g_1, g_2, h \in G$, we have*

$$g_1 \geq g_2 \ \Rightarrow \ g_1 \cdot h \geq g_2 \cdot h \quad \textit{and} \quad h \cdot g_1 \geq h \cdot g_2. \tag{1}$$

Let $(R, \leq)$ be a real closed field. Consider its subset

$$Pos(R) := \{x \in R \mid 0 < x\}.$$

a) Show that $(Pos(R), \cdot)$ is an ordered abelian subgroup of $(R^\times, \cdot)$.

b) Show that $(Pos(R), \cdot)$ is *divisible*, i.e. for any $a \in Pos(R)$, we have $a^q \in Pos(R)$ for all $q \in \mathbb{Q}$.

3) Let R be a real closed field. Recall that:

Definition 2.
Given $m, n \in \mathbb{N}$, let $A \subset R^m$ and $B \subset R^n$ be two semi-algebraic sets. A mapping $f : A \to B$ is said to be semi-algebraic *if its graph*

$$\Gamma_f := \{(\underline{x}, \underline{y}) \in R^{m+n} \mid \underline{y} = f(\underline{x})\}$$

is a semi-algebraic subset of R^{m+n}.

The aim of this exercise is to prove that the real exponential function exp is not semi-algebraic.

a) Consider some polynomials $p_0, \ldots, p_n \in \mathbb{R}[X]$, and an infinite subset $U \subset \mathbb{R}$ s.t. for all $x \in U$

$$p_n(x)(e^x)^n + p_{n-1}(x)(e^x)^{n-1} + \cdots + p_0(x) = 0.$$

Show that $p_0 \equiv \cdots \equiv p_n \equiv 0$ in the following way:

i) if U has no bound, then use just the comparison between exponential and polynomials .

ii) if U is bounded, then use the following result.

Theorem 3 (Identity Theorem of Complex Analysis)**.**
Consider two complex functions $f(z)$ and $g(z)$ holomorphic (i.e. differentiable with respect to their complex variable z) in a domain D of the complex plane $\mathbb{C}$. If the equation $f(z) = g(z)$ holds in an infinite subset S of D having an accumulation point in D, then it holds in the whole D.

b) Show by contradiction that exp is not semi-algebraic.
[Hint: show that $\Gamma_{\exp}$ would have a non-empty interior using the normal form for semi-alg. sets.]

4) We consider the *Motzkin polynomial*

$$m(X, Y) := 1 - 3X^2Y^2 + X^2Y^4 + X^4Y^2.$$

a) Show that $m(X, Y) \geq 0$ on $\mathbb{R}^2$.
[Hint: use the following inequality which relates the arithmetic mean to the geometric mean: $\forall\, 0 \leq a, b, c \in \mathbb{R}, \quad \frac{a+b+c}{3} \geq \sqrt[3]{abc}$.]

b) Suppose that the Motzkin polynomial $m = f_1^2 + \cdots + f_k^2$ for some $k \in \mathbb{N}$ and $f_i(X, Y) \in \mathbb{R}[X, Y]$. Deduce that for any $i \in \{1, \ldots, k\}$, $f_i(X, Y)$ is a polynomial of degree at most 3 and $f_i(X, Y)$ can contain none of the following monomials: X^3, Y^3, X^2, Y^2, X and Y.

c) Conclude by contradiction that m cannot be a sum of squares in $\mathbb{R}[X, Y]$.
[Hint: for any $i \in \{1, \ldots, k\}$ write $f_i(X, Y) = a_i + b_iXY + c_iX^2Y + d_iXY^2$.]

5) Let $n \in \mathbb{N}$, $\underline{X} := (X_1, \ldots, X_n)$ and $d \in \mathbb{N}_0$. Consider some non-zero polynomial $f \in \mathbb{R}[\underline{X}]$ with $\deg(f) \leq d$.

a) Show that

$$\overline{f}(X_0, X_1, \ldots, X_n) := X_0^d f\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right)$$

is a homogenous polynomial (or form) of degree $= d$ in the variables $X_0, X_1 \ldots, X_n$.

Definition 4.
The polynomial $\overline{f}(X_0, X_1, \ldots, X_n)$ *is said to be the* ***homogenization*** *of* $f(X_1, \ldots, X_n)$.

b) Denote by $V_{d,n}$ the $\mathbb{R}$-vector space of all polynomials in $\mathbb{R}[X_1, \ldots, X_n]$ of degree $\leq d$, and by $F_{d,n+1}$ the $\mathbb{R}$-vector space of all homogenous polynomials in $\mathbb{R}[X_0, X_1, \ldots, X_n]$ of degree $= d$. Show that the homogenization map

$$\begin{array}{rccc} h: & V_{d,n} & \to & F_{d,n+1} \\ & f(X_1, \ldots, X_n) & \mapsto & h(f) := \overline{f}(X_0, X_1, \ldots, X_n) \end{array}$$

is a vector space isomorphism between $V_{d,n}$ and $F_{d,n+1}$.

c) From now on, we suppose that d is even. Show that $f \geq 0$ on $\mathbb{R}^n$ if and only if $\overline{f} \geq 0$ on $\mathbb{R}^{n+1}$.

d) Show that f is a sum of squares of polynomials in $\mathbb{R}[X_1, \ldots, X_n]$ if and only if $\overline{f}$ is a sum of squares of homogenous polynomials in $\mathbb{R}[X_0, X_1, \ldots, X_n]$ of degree $d/2$. (Recall that we are assuming $\deg(f) \leq d$ and d even.)

e) Show that the homogenous Motzkin polynomial

$$\overline{m}(X, Y, Z) = Z^6 + X^4Y^2 + X^2Y^4 - 3X^2Y^2Z^2$$

is positive semi-definite on $\mathbb{R}^3$ but is not a sum of squares.