Universität Konstanz
Fachbereich Mathematik und Statistik
Prof. Dr. Salma Kuhlmann
Dr. Maria Infusino
Dr. Charu Goel

REAL ALGEBRAIC GEOMETRY–WS 2014/15
Exercise Sheet 1

This assignment is due by Tuesday the 28th of October by noon. Your solutions will be collected during Tuesday's lecture or you can drop them in the postbox 18 near F411.

1. Let $X = \{a, b\}$. For every $A, B \subseteq X$, consider the relation
$$A \leq B \Leftrightarrow A \subseteq B.$$
   (a) Show that $\leq$ is a partial order on the power set $\mathcal{P}(X)$ which is NOT total. Recall that $\mathcal{P}(X)$ is the collection of all possible subset of X.
   (b) What happens if we consider $\leq$ on $Y = \{\emptyset, \{a\}, \{a, b\}\}$? Is $\leq$ still a partial order? Is it total?

2. Let $(K, \leq)$ be a totally ordered field. For any $a, b, c \in K$, show that:
   (a) $a \leq 0 \Leftrightarrow -a \geq 0$
   (b) $0 \leq a^2$
   (c) $a \leq b$, $0 \leq c \Rightarrow ac \leq bc$
   (d) $0 < a \leq b \Leftrightarrow 0 < b^{-1} \leq a^{-1}$
   (e) $0 < ab \Leftrightarrow 0 < \frac{a}{b}$
   (f) $0 < n$, $\forall n \in \mathbb{N}$
   (g) $\text{sign}(ab) = \text{sign}(a)\,\text{sign}(b)$.
   (h) If $|a| > |b|$ then $\text{sign}(a + b) = \text{sign}(a)$.

3. Let $(K, \leq)$ a totally ordered field. Prove that the following are equivalent
   (a) $(K, \leq)$ is Archimedean
   (b) $\mathbb{N}$ is cofinal in K
   (c) $-\mathbb{N}$ is coinitial in K
   (d) $\mathbb{Z}$ is coterminal in K.

4. Let $(K, \leq)$ be a totally ordered set. Show that:
   (a) $(K, \leq)$ is Dedekind complete if and only if $(K, \leq)$ has no free Dedekind cuts.
   (b) If $(K, \leq)$ is an ordered field which is Dedekind complete, then $(K, \leq)$ is Archimedean.