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REAL ALGEBRAIC GEOMETRY–WS 2014/15

Exercise Sheet 10

This assignment is due by Tuesday the 20th of January at noon.
Your solutions will be collected during Tuesday's lecture or you can drop them in the postbox 18 near F411.

1) Let $n \in \mathbb{N}$, $n \geq 1$. For $0 \not\equiv f \in \mathbb{R}[x_1, \ldots, x_n]$, let

$$\mathcal{Z}(f) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ f(x_1, \ldots, x_n) = 0\}.$$

Prove that $\mathbb{R}^n \setminus \mathcal{Z}(f)$ is dense in $\mathbb{R}^n$.

Is it still true replacing $\mathbb{R}$ by any real closed field R?

2) Let A be a commutative ring with 1. We recall that :

- $T \subseteq A$ is a *preordering* of A if
  $1 \in T$, $T + T \subseteq T$, $\forall a \in A$, $a^2 \cdot T \subseteq T$ and $T \cdot T \subseteq T$.
- A preordering P is an *ordering* of A if
  $A = -P \cup P$ and $-P \cap P$ is a prime ideal of A.

Let A be the ring of continuous functions $f : [0; 1] \to \mathbb{R}$. Find a preordering T and an ordering P of A such that the following conditions are satisfied:

a) $\sum A^2 \subsetneq T \subsetneq P$

b) there are infinitely many preorderings T_i such that $\sum A^2 \subsetneq T_i \subsetneq T$

c) there are infinitely many preorderings S_i such that $T \subsetneq S_i \subsetneq P$

3) Let $n \in \mathbb{N}$, K be a field and V be an algebraic subset of K^n.

a) Show that:

- **i)** $\mathcal{I}(V)$ is an ideal of $K[x_1, \ldots, x_n]$.
- **ii)** $\mathcal{Z}(\mathcal{I}(V)) = V$.
- **iii)** the map $V \to \mathcal{I}(V)$ is an injection from the set of algebraic subsets of K^n into the set of ideals of $K[x_1, \ldots, x_n]$.

b)
- **i)** Show that for any ideal $I \subseteq K[x_1, \ldots, x_n]$ the inclusion $I \subseteq \mathcal{I}(\mathcal{Z}(I))$ is always true.
- **ii)** Give an example of ideal $I \subseteq K[x_1, \ldots, x_n]$ such that $I \subsetneq \mathcal{I}(\mathcal{Z}(I))$.

4) Let A be a commutative ring with 1 and $I \subseteq A$ an ideal. We recall that:

- I is *prime* if I is proper and ($ab \in I \Rightarrow a \in I$ or $b \in I$),
- I is *radical* if $I = \sqrt{I} := \{a \in A : \exists m \in \mathbb{N} \text{ s.t. } a^m \in I\}$,
- I is *real* if $I = \sqrt[R]{I} := \{a \in A : \exists m \in \mathbb{N}, \exists \sigma \in \sum A^2 \text{ s.t. } a^{2m} + \sigma \in I\}$.

a) Show that any prime ideal is radical.

b) Give an example of an ideal $I \subseteq K[x_1, \ldots, x_n]$ (for some field K and some $n \in \mathbb{N}$) which is radical but not prime.

c) Give an example of an ideal $I \subseteq K[x_1, \ldots, x_n]$ (for some field K and some $n \in \mathbb{N}$) which is prime but not real.

Please, justify your answers!