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REAL ALGEBRAIC GEOMETRY–WS 2014/15

Exercise Sheet 11

This assignment is due by Tuesday the 27th of January at noon.
Your solutions will be collected during Tuesday's lecture or you can drop them in the postbox 18 near F411.

1) a) Let f be a homogeneous polynomial in $\mathbb{R}[\underline{x}]$. Show that if f is sum of squares then every sum of square representation of f consists of homogeneous polynomials, namely:

f is homogeneous, $f = f_1^2 + \cdots + f_s^2 \;\Rightarrow\; f_i$ is homogeneous $\forall\, i = 1, \ldots, s$.

b) Show that

$$f \in \mathcal{P}_{n,2d}, \quad f = f_1^2 + \cdots + f_s^2 \;\Rightarrow\; \exists \text{ such an } s \text{ with } s \leq \binom{n+d}{d}.$$

2) a) Let $F(\mathrm{x}, \mathrm{y}) := \mathrm{x}^6 + \mathrm{x}^4\mathrm{y}^2 + 3\mathrm{x}^2\mathrm{y}^4 + 3\mathrm{y}^6$. Write $F(\mathrm{x}, \mathrm{y})$ as a sum of two squares.

b) Let $G(\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{t}) := 2\mathrm{x}^2 + 2\mathrm{xy} + 2\mathrm{y}^2 + 3\mathrm{z}^2 + 2\mathrm{zt} + 3\mathrm{t}^2$. Write $G(\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{t})$ as a sum of four squares.

3) Let R be a real closed field. We denote by $\mathcal{P}_{n,m}(R)$ the set of psd forms with coefficients in R of degree m in n variables, and with $\Sigma_{n,m}(R)$ the set of forms with coefficients in R of degree m in n variables which are sums of squares. Show that :

a) for every $d \in \mathbb{N}$, $\mathcal{P}_{2,2d}(R) = \Sigma_{2,2d}(R)$.
b) for every $n \in \mathbb{N}$, $\mathcal{P}_{n,2}(R) = \Sigma_{n,2}(R)$.
c) $\mathcal{P}_{3,4}(R) = \Sigma_{3,4}(R)$.
Use that Hilbert proved that:

$(*) \qquad f \in \mathcal{P}_{3,4}(\mathbb{R}) \;\Rightarrow\; \exists\, f_1, f_2, f_3 \in \mathcal{F}_{3,2}(\mathbb{R})$ such that $f = f_1^2 + f_2^2 + f_3^2$.

4) Show that $\forall\, n \in \mathbb{N}$ and $\forall\, \alpha_1, \ldots, \alpha_n, x_1, \ldots, x_n \in \mathbb{R}^{\geq 0} = \{y \in \mathbb{R} : y \geq 0\}$

$$\sum_{i=1}^{n} \alpha_i = 1 \;\Rightarrow\; \alpha_1 x_1 + \cdots + \alpha_n x_n - x_1^{\alpha_1} \cdots x_n^{\alpha_n} \geq 0.$$

Please, justify your answers!