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# REAL ALGEBRAIC GEOMETRY–WS 2014/15

## Exercise Sheet 12

*This assignment is due by Tuesday the 3rd of February at noon.*
*Your solutions will be collected during Tuesday's lecture or you can drop them in the postbox 18 near F411.*

**1)** Let $K$ be a real closed field and suppose that $0 \not\equiv f \in K[x_1, \dots, x_n]$ is irreducible. Show that if $f$ changes sign on $K^n$ (i.e. $\exists\, x, y \in K^n$ s.t. $f(x)f(y) < 0$) then $(f) = \mathcal{I}(\mathcal{Z}(f))$, where $(f)$ is the principal ideal generated by $f$ and $\mathcal{I}(\mathcal{Z}(f))$ is the ideal of vanishing polynomials on the zero set of $f$.

**N.B.** This argument was used for $f$ linear and $n = 2$ in the proof of Case 2.2 in Lemma 1.2 of Lecture 8 in the script on Positive Polynomials from SS10.

**2)** Show that the symmetric quaternary quartic:

$$F(x_1, x_2, x_3, x_4) := \sum_{\substack{i,j=1 \\ i<j}}^{4} x_i^2 x_j^2 + \sum_{\substack{i,j,k=1 \\ i\neq j\neq k \\ j<k}}^{4} x_i^2 x_j x_k - 2x_1x_2x_3x_4$$

is psd but not sos.

[Hint: recall that $Q(x, y, z, w) := w^4 + x^2y^2 + y^2z^2 + z^2x^2 - 4xyzw \in \mathcal{P}_{4,4} \backslash \Sigma_{4,4}$ (see Prop. 4.2 of Lecture 9 in the script on Positive Polynomials from SS10.)]

**3)** Let $A$ be a commutative ring with 1 and let

$$\chi := \mathrm{Hom}(A, \mathbb{R}) = \{\alpha \colon A \to \mathbb{R} \mid \alpha \text{ is a ring homomorphism}\}.$$

Consider the map defined by

$$\begin{aligned} \chi &\to \operatorname{Sper} A \\ \alpha &\mapsto P_\alpha := \alpha^{-1}\left(\mathbb{R}^{\geq 0}\right), \end{aligned} \tag{1}$$

where $Sper(A) := \{P : P \text{ is an ordering of} A\}$.

Show that:

**a)** this map is well-defined, i.e. $P_\alpha \subseteq A$ is an ordering;

**b)** this map is injective, i.e. $\alpha \neq \beta \Rightarrow P_\alpha \neq P_\beta$;

**c)** $\operatorname{supp}(P_\alpha) = \ker \alpha$

For every $a \in A$, define:

$$\begin{aligned} \hat{a} \colon \chi &\to \mathbb{R} \\ \alpha &\mapsto \hat{a}(\alpha) := \alpha(a) \end{aligned} \tag{2}$$

and

$$\mathcal{U}(\hat{a}) := \{\alpha \in \chi \mid \hat{a}(\alpha) > 0\}.$$

Show that:

**d)** the collection $\mathcal{B} := \{\, \mathcal{U}(\hat{a}) \mid a \in A\}$ is a sub-base for a topology $\tau$ on $\chi$;

**e)** for every $a \in A$ the map $\hat{a} \colon \chi \to \mathbb{R}$ defined in (2) is continuous with the respect to the topology $\tau$;

**f)** if $\tau_1$ is another topology on $\chi$ such that $\hat{a}$ is continuous for every $a \in A$, then $\mathcal{U}(\hat{a}) \in \tau_1$ for every $a \in A$ (i.e. $\tau_1$ has more open sets than $\tau$).

[In other words, the topology $\tau$ is the *weakest topology* on $\chi$ for which the map $\hat{a}$ is continuous for every $a \in A$.]

**g)** if we endow $\operatorname{Sper} A$ with the spectral topology, then the topology induced by the map in (1) coincide with $\tau$.

[Recall that a sub-basis of open sets for the spectral topology is given by the collection $\{u(a) \colon a \in A\}$ where $u(a) := \{P \in \operatorname{Sper} A \colon a \notin -P\}$.]

**4)** Let $A$ be a commutative ring with 1 containing $\mathbb{Q}$. Let $T$ be a generating preprime and $M$ a maximal proper $T$-module. Suppose $M$ is Archimedean. Define the map

$$\begin{aligned} \alpha \colon A &\longrightarrow \mathbb{R} \\ a &\mapsto \inf\{r \in \mathbb{Q} : r - a \in M\}. \end{aligned}$$

Show that $\alpha$ is a ring homomorphism.