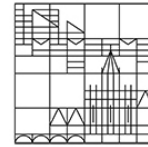


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## REAL ALGEBRAIC GEOMETRY–WS 2014/15

### Exercise Sheet 12

*This assignment is due by Tuesday the 3rd of February at noon.  
 Your solutions will be collected during Tuesday's lecture or you can drop them  
 in the postbox 18 near F411.*

- 1) Let  $K$  be a real closed field and suppose that  $0 \neq f \in K[x_1, \dots, x_n]$  is irreducible. Show that if  $f$  changes sign on  $K^n$  (i.e.  $\exists x, y \in K^n$  s.t.  $f(x)f(y) < 0$ ) then  $(f) = \mathcal{I}(\mathcal{Z}(f))$ , where  $(f)$  is the principal ideal generated by  $f$  and  $\mathcal{I}(\mathcal{Z}(f))$  is the ideal of vanishing polynomials on the zero set of  $f$ .

**N.B.** This argument was used for  $f$  linear and  $n = 2$  in the proof of Case 2.2 in Lemma 1.2 of Lecture 8 in the script on Positive Polynomials from SS10.

- 2) Show that the symmetric quaternary quartic:

$$F(x_1, x_2, x_3, x_4) := \sum_{\substack{i,j=1 \\ i < j}}^4 x_i^2 x_j^2 + \sum_{\substack{i,j,k=1 \\ i \neq j \neq k \\ j < k}}^4 x_i^2 x_j x_k - 2x_1 x_2 x_3 x_4$$

is psd but not sos.

[Hint: recall that  $Q(x, y, z, w) := w^4 + x^2 y^2 + y^2 z^2 + z^2 x^2 - 4xyzw \in \mathcal{P}_{4,4} \setminus \Sigma_{4,4}$  (see Prop. 4.2 of Lecture 9 in the script on Positive Polynomials from SS10.)]

- 3) Let  $A$  be a commutative ring with 1 and let

$$\chi := \text{Hom}(A, \mathbb{R}) = \{\alpha : A \rightarrow \mathbb{R} \mid \alpha \text{ is a ring homomorphism}\}.$$

Consider the map defined by

$$\begin{aligned} \chi &\rightarrow \text{Sper } A \\ \alpha &\mapsto P_\alpha := \alpha^{-1}(\mathbb{R}^{\geq 0}), \end{aligned} \tag{1}$$

where  $\text{Sper}(A) := \{P : P \text{ is an ordering of } A\}$ .

Show that:

- a) this map is well-defined, i.e.  $P_\alpha \subseteq A$  is an ordering;
- b) this map is injective, i.e.  $\alpha \neq \beta \Rightarrow P_\alpha \neq P_\beta$ ;

c)  $\text{supp}(P_\alpha) = \ker \alpha$

For every  $a \in A$ , define:

$$\begin{aligned} \hat{a}: \chi &\rightarrow \mathbb{R} \\ \alpha &\mapsto \hat{a}(\alpha) := \alpha(a) \end{aligned} \tag{2}$$

and

$$\mathcal{U}(\hat{a}) := \{\alpha \in \chi \mid \hat{a}(\alpha) > 0\}.$$

Show that:

- d) the collection  $\mathcal{B} := \{\mathcal{U}(\hat{a}) \mid a \in A\}$  is a sub-base for a topology  $\tau$  on  $\chi$ ;
  - e) for every  $a \in A$  the map  $\hat{a}: \chi \rightarrow \mathbb{R}$  defined in (2) is continuous with respect to the topology  $\tau$ ;
  - f) if  $\tau_1$  is another topology on  $\chi$  such that  $\hat{a}$  is continuous for every  $a \in A$ , then  $\mathcal{U}(\hat{a}) \in \tau_1$  for every  $a \in A$  (i.e.  $\tau_1$  has more open sets than  $\tau$ ).  
[In other words, the topology  $\tau$  is the *weakest topology* on  $\chi$  for which the map  $\hat{a}$  is continuous for every  $a \in A$ .]
  - g) if we endow  $\text{Sper } A$  with the spectral topology, then the topology induced by the map in (1) coincide with  $\tau$ .  
[Recall that a sub-basis of open sets for the spectral topology is given by the collection  $\{u(a) : a \in A\}$  where  $u(a) := \{P \in \text{Sper } A : a \notin -P\}$ .]
- 4) Let  $A$  be a commutative ring with 1 containing  $\mathbb{Q}$ . Let  $T$  be a generating preprime and  $M$  a maximal proper  $T$ -module. Suppose  $M$  is Archimedean. Define the map

$$\begin{aligned} \alpha: A &\longrightarrow \mathbb{R} \\ a &\mapsto \inf\{r \in \mathbb{Q} : r - a \in M\}. \end{aligned}$$

Show that  $\alpha$  is a ring homomorphism.