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REAL ALGEBRAIC GEOMETRY–WS 2014/15
Exercise Sheet 2

This assignment is due by Tuesday the 4th of November at noon.
Your solutions will be collected during Tuesday's lecture or you can drop them in the postbox 18 near F411.

1) Recall the following definition:

Definition 1. *An ordered field $(K, \leq)$ is:*

- ***Dedekind complete**, if for any pair of non-empty subsets L and U of K such that $L \leq U$ (i.e. $\lambda \leq \mu$ for any $\lambda \in L$ and any $\mu \in U$), there exists $\alpha \in K$ such that $L \leq \alpha \leq U$;*
- ***Archimedean** if for any $\alpha \in K$, there exists $n \in \mathbb{N}$ such that $\alpha < n$.*

a) Show that if $(K, \leq)$ is an archimedean ordered field, then $\mathbb{Q}$ is dense in $(K, \leq)$, i.e. $\forall x, y \in K$ with $x < y$, $\exists\, q \in \mathbb{Q}$, $x < q < y$.

b) Let $(K, \leq)$ be an archimedean ordered field and let $\varphi : K \to \mathbb{R}$ be the map defined in the proof the Hölder theorem introduced in Lecture 2, i.e. for any $a \in K$, $\varphi(a) := \sup I_a = \inf U_a \in \mathbb{R}$, where:

$$I_a := \{r \in \mathbb{Q} \mid r \leq a\} \text{ and } U_a := \{r \in \mathbb{Q} \mid a \leq r\}.$$

Show that:

(i) φ is a ring homomorphism between K and $\mathbb{R}$, and therefore a field embedding;

(ii) φ preserves the order, i.e. if $a, b \in K$, $a \leq b$ then $\varphi(a) \leq \varphi(b)$.

c) Prove that $(\mathbb{R}, \leq)$ is the unique Dedekind complete ordered field up to isomorphism. Namely, show that:

(i) $(\mathbb{R}, \leq)$ is a Dedekind complete ordered field.

(ii) Let $(K, \leq)$ be a Dedekind complete ordered field. Then K is isomorphic to $\mathbb{R}$ as ordered field.

[Hint: recall Exercise 4 in Sheet 1]

2) Recall the following definition:

Definition 2. *A* ***preordering*** *or* ***cone*** *of a field K is a subset P of K s.t.*

(1) $a, b \in P \Rightarrow a + b \in P$;

(2) $a, b \in P \Rightarrow a \cdot b \in P$;

(3) $a \in K \Rightarrow a^2 \in P$.

A cone P of K is said to be ***proper*** *if in addition to (1), (2), (3) the following property holds:*
(4) $-1 \notin P$.

A cone P of K is said to be ***positive*** *if in addition to (1), (2), (3), (4) the following property holds:*
(5) $-P \cup P = K$, *where* $-P := \{-a \in K \mid a \in P\}$.

a) Show that a cone P in a field K is proper if and only if $-P \cap P = \{0\}$.

b) Prove that:

(i) If $(K, \leq)$ is an ordered field, then the subset $P := \{a \in K \mid a \geq 0\}$ is a positive cone of K.

(ii) Conversely, if P is a positive cone of a field K, then the relation

$$a \leq b \Leftrightarrow b - a \in P$$

defines an order on K such that $(K, \leq)$ is an ordered field.

c) Deduce that, for any field K, there is a bijective correspondence between the set of orderings on K and the set of positive cones of K.

3) Recalling that the **set of sums of squares** of elements of a field K is denoted by $\sum K^2$, show that:

a) $\sum K^2$ is a cone.

b) $\sum K^2$ is the smallest cone of K.

c) If K is (formally) real, then $\sum K^2$ is a proper cone.

d) If K is algebraically closed, then K is not real.

e) If (K, P) is an ordered field, F is another field and $\varphi : F \to K$ a field homomorphism, then $Q := \varphi^{-1}(P)$ is an ordering of F. In this case, we say that P is an **extension** of Q and Q is the **pullback** of P.

f) If P_1 and P_2 are positive cones of K with $P_1 \subseteq P_2$ then $P_1 = P_2$.

g) In particular, if $\sum K^2$ is a positive cone of K, then it is the only ordering of K.

h) The fields $\mathbb{R}$ and $\mathbb{Q}$ admit a unique ordering.