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REAL ALGEBRAIC GEOMETRY–WS 2014/15

Exercise Sheet 3

This assignment is due by Tuesday the 11th of November at noon.
Your solutions will be collected during Tuesday's lecture or you can drop them in the postbox 18 near F411.

1) Let $\mathcal{T} = \{T_i : i \in I\}$ be a family of preorderings on a field K. Prove that:

a) the intersection $\bigcap_{i \in I} T_i$ is a preordering of K,

b) if $\forall\, i, j \in I$ there exists $k \in I$ such that $T_i \cup T_j \subseteq T_k$, then $\bigcup_{i \in I} T_i$ is a preordering of K.

2) Show that any ordering on a field K extends to an order on the field of rational functions in several variables $K(x_1, ..., x_n)$, using induction on n.

[Hint: Prove the statement by contradiction using the criterion for extension of orderings in the case of a field extension L/K, given in Lecture 4.]

3) We proved in Lecture 2 that each Dedekind cut of $\mathbb{R}$ corresponds to an ordering on $\mathbb{R}[x]$, in particular on $\mathbb{R}(x)$. Describe explicitly the order on $\mathbb{R}[x]$ corresponding to each Dedekind cut of $\mathbb{R}$. Proceed as follows.

a) Retrieve the orders on $\mathbb{R}[x]$ corresponding to the cuts 0_+ and 0_-, using the derivatives of a generic polynomial $p \in \mathbb{R}[x]$ at 0.

b) Using the same technique as in a), describe the orders on $\mathbb{R}[x]$ corresponding to all the remaining Dedekind cuts of $\mathbb{R}$.

c) Conclude that there exists a function

$$\sigma : \mathbb{R}[x] \to \mathbb{R}$$

such that $\operatorname{sign}(p(x)) = \operatorname{sign}(\sigma(p))$ for any $p \in \mathbb{R}[x]$.

4) Denote the ring of real formal power series in one variable by:

$$\mathbb{R}[[X]] \quad := \quad \left\{ \sum_{i=0}^{\infty} a_i X^i \mid a_i \in \mathbb{R} \right\},$$

and the set of **real formal Laurent series** by:

$$\mathbb{R}((X)) := \left\{ \sum_{i=m}^{\infty} a_i X^i \mid m \in \mathbb{Z},\ a_i \in \mathbb{R} \right\}.$$

For any $0 \neq A \in \mathbb{R}((X))$, we define $v(A)$ to be the smallest integer m such that $a_m \neq 0$. Moreover, for any $A(X) = \sum_{i=m}^{\infty} a_i X^i \in \mathbb{R}((X))$ and $B(X) = \sum_{i=n}^{\infty} b_i X^i \in \mathbb{R}((X))$ with $m = v(A)$ and $n = v(B)$, we define:

- the **coefficientwise addition**:

$$A(X) + B(X) := \sum_{i=k}^{\infty} (a_i + b_i) X^i, \text{ where } k = \min\{v(A), v(B)\}$$

- the **convolution product**:

$$A(X)B(X) := \sum_{i=m+n}^{\infty} \left(\sum_{j+k=i} a_j b_k \right) X^i$$

- the **order relation**:
  For any $0 \neq A(X) = \sum_{i=m}^{\infty} a_i X^i \in \mathbb{R}((X))$

$$A(X) > 0 \text{ if and only if } a_m > 0 \text{ with } m = v(A).$$

It can be shown that $\mathbb{R}((X))$ endowed with these relations is an ordered field. Note that the map $v : \mathbb{R}((X))^{\times} \to \mathbb{Z}$ is a **discrete valuation** on $\mathbb{R}((X))^{\times} := \mathbb{R}((X)) \setminus \{0\}$, i.e. v is such that for any $A, B \in \mathbb{R}((X))^{\times}$ the following hold:

i) $v(A + B) \geq \min\{v(A), v(B)\}$,

ii) $v(AB) = v(A) + v(B)$.

Then prove that $\mathbb{R}((X))$ is not real closed.

[Hint: Use the property ii) of the map v on $\mathbb{R}((X))^{\times}$.]