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REAL ALGEBRAIC GEOMETRY–WS 2014/15

Exercise Sheet 4

This assignment is due by Tuesday the 18th of November at noon.
Your solutions will be collected during Tuesday's lecture or you can drop them in the postbox 18 near F411.

1) Let R be a real closed field and let $f(x) = dx^m + d_{m-1}x^{m-1} + \cdots + d_0$ be a polynomial in $R[x]$ with $d \neq 0$. Show that the following statements are equivalent:

a) $f \geq 0$ on R (i.e. $f(x) \geq 0$ for any $x \in R$),

b) $d > 0$ and all the real roots of f have even multiplicity,

c) $f = g^2 + h^2$ for some $g, h \in R[x]$.

2) Let R be a real closed field and let $f(x) = x^m + d_{m-1}x^{m-1} + \cdots + d_0$ be a monic polynomial in $R[x]$, whose roots $a_1, \ldots, a_m$ are all reals. Show that:

$$a_i \geq 0 \text{ for all } i = 1, \ldots, m \ \Leftrightarrow\ (-1)^{m-i}d_i \geq 0 \text{ for all } i = 0, \ldots, m-1.$$

3) Construct a *countable* Archimedean real closed field, proceeding as follows:

a) Show that the field of real numbers is real closed.

b) Let R be real closed, K a subfield of R and $\tilde{K}^r$ the relative algebraic closure of K in R, i.e. the set of all elements of R which are algebraic over K. Show that $\tilde{K}^r$ is real closed.

c) Consider the relative algebraic closure $\tilde{\mathbb{Q}}^r$ of $\mathbb{Q}$ in $\mathbb{R}$, i.e. the field of real algebraic numbers. Show that $\tilde{\mathbb{Q}}^r$ is a countable real closed field.

4) Construct a countable Archimedean not real closed field which is a proper extension of $\mathbb{Q}$.

Please, justify your answers!