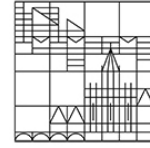


Universität Konstanz
Fachbereich Mathematik und Statistik
Prof. Dr. Salma Kuhlmann
Dr. Maria Infusino
Dr. Charu Goel



REAL ALGEBRAIC GEOMETRY–WS 2014/15

Exercise Sheet 5

*This assignment is due by Tuesday the 25th of November at noon.
Your solutions will be collected during Tuesday's lecture or you can drop them
in the postbox 18 near F411.*

- 1) Prove that there are 2^{\aleph_0} pairwise distinct Archimedean orderings on $\mathbb{Q}(x)$.
Recall that 2^{\aleph_0} is the cardinality of the continuum.
- 2) Let R be a real closed field and let $f(x) = x^3 + 6x^2 - 16$ in $R[x]$.
 - a) Compute the Sturm sequence of $f(x)$.
 - b) Show that $f(x)$ has three distinct roots in R .
 - c) Denote them by $\alpha_1 < \alpha_2 < \alpha_3$. Show that for all i , $\alpha_i \in [-7, 2]$, and that $\alpha_1 \in [-6, -5]$, $\alpha_2 = -2$ and that $\alpha_3 \in [1, 2]$.
- 3) Consider the ring of real formal power series

$$\mathbb{R}[[X]] := \left\{ \sum_{i=0}^{\infty} a_i X^i \mid a_i \in \mathbb{R} \right\}$$

and the field of real Laurent series

$$\mathbb{R}((X)) := \left\{ \sum_{i=m}^{\infty} a_i X^i \mid m \in \mathbb{Z}, a_i \in \mathbb{R} \right\}.$$

(see Exercise Sheet 3). Show that $\mathbb{R}((X))$ admits exactly two orderings extending the one on \mathbb{R} and that they are both non-Archimedean.

[Hint: show that formal power series $1 + \sum_{i=1}^{\infty} a_i X^i$ are squares]

- 4) Prove that $L := \mathbb{Q}(\sqrt{2} + \sqrt{3})$ has exactly four orderings P_1, P_2, P_3, P_4 that extend the one on \mathbb{Q} . Moreover, show explicitly that the orderings are distinct, i.e. $P_i \neq P_j$ for any $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$.

Please, justify your answers!