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# REAL ALGEBRAIC GEOMETRY–WS 2014/15

## Exercise Sheet 5

*This assignment is due by Tuesday the 25th of November at noon.*
*Your solutions will be collected during Tuesday's lecture or you can drop them in the postbox 18 near F411.*

**1)** Prove that there are $2^{\aleph_0}$ pairwise distinct Archimedean orderings on $\mathbb{Q}(x)$. Recall that $2^{\aleph_0}$ is the cardinality of the continuum.

**2)** Let $R$ be a real closed field and let $f(x) = x^3 + 6x^2 - 16$ in $R[x]$.

**a)** Compute the Sturm sequence of $f(x)$.

**b)** Show that $f(x)$ has three distinct roots in $R$.

**c)** Denote them by $\alpha_1 < \alpha_2 < \alpha_3$. Show that for all $i$, $\alpha_i \in [-7, 2]$, and that $\alpha_1 \in [-6, -5]$, $\alpha_2 = -2$ and that $\alpha_3 \in [1, 2]$.

**3)** Consider the ring of real formal power series

$$\mathbb{R}[[X]] := \left\{ \sum_{i=0}^{\infty} a_i X^i \mid a_i \in \mathbb{R} \right\}$$

and the field of real Laurent series

$$\mathbb{R}((X)) := \left\{ \sum_{i=m}^{\infty} a_i X^i \mid m \in \mathbb{Z},\ a_i \in \mathbb{R} \right\}.$$

(see Exercise Sheet 3). Show that $\mathbb{R}((X))$ admits exactly two orderings extending the one on $\mathbb{R}$ and that they are both non-Archimedean.

[Hint: show that formal power series $1 + \sum_{i=1}^{\infty} a_i X^i$ are squares]

**4)** Prove that $L := \mathbb{Q}(\sqrt{2} + \sqrt{3})$ has exactly four orderings $P_1, P_2, P_3, P_4$ that extend the one on $\mathbb{Q}$. Moreover, show explicitly that the orderings are distinct, i.e. $P_i \neq P_j$ for any $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$.

***Please, justify your answers!***