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REAL ALGEBRAIC GEOMETRY–WS 2014/15

Exercise Sheet 6

This assignment is due by Tuesday the 2nd of December at noon.
Your solutions will be collected during Tuesday's lecture or you can drop them in the postbox 18 near F411.

1) Let $(K,\leq)$ be an ordered field which verifies the intermediate value property:

$$\forall a<b\in K,\ f(a)<0<f(b)\ \Rightarrow\ \exists\, c\in]a,b[,\ f(c)=0.$$

Show that K is real closed.

2) Let R be a real closed field and $R(i)$ be its algebraic closure (with $i:=\sqrt{-1}$). Consider the following system consisting of one cubic equation:

$$S(\underline{T};X):=\begin{cases}X^3+T_1X^2+T_2X+T_3=0.\end{cases} \tag{1}$$

The aim of this exercise is to explicitly show how to eliminate the variable X in $S(\underline{T},X)$. We will adapt to our context the classical *method of Cardano* of resolution of cubic equations in one variable, in order to write down explicitly the systems $S_1(\underline{T}),S_2(\underline{T}),S_3(\underline{T})$ whose existence is guaranteed by the basic version of the Tarski-Seidemberg Theorem applied to $S(\underline{T},\underline{X})$.

a) Show that (1) is equivalent to

$$\tilde{S}(Z,W;Y):=\begin{cases}Y^3+ZY+W=0.\end{cases} \tag{2}$$

where Y,Z and W depend themselves polynomially on T_1,T_2,T_3,X. [Hint: use the Tschirnhausen transform $X=Y-\dfrac{T_1}{3}$.]

b) Use the changes of variables

$$Y=U+V \text{ and } Z=-3UV$$

to show that, for any $w,z\in R$, there exist $u,v\in R[i]$ such that u^3 and v^3 are the two solutions with respect to the variable H of the following

$$\begin{cases}H^2+wH-\dfrac{z^3}{27}=0.\end{cases} \tag{3}$$

c) Define $\Delta(Z,W) := W^2 + \dfrac{4}{27}Z^3$ (the so-called *discriminant* of the cubic equation (2)) and consider $w, z \in R$. Deduce that:

(i) if $\Delta(z,w) > 0$, then $\tilde{S}(z,w;Y)$ defined in (2) has 1 real solution

(ii) if $\Delta(z,w) = 0$, then $\tilde{S}(z,w;Y)$ defined in (2) has 2 real solutions

(iii) if $\Delta(z,w) < 0$, then $\tilde{S}(z,w;Y)$ defined in (2) has 3 real solutions

and give explicit formulas (the so-called *Cardano formulas*) for such solutions, which clearly depend on z, w.

[Recall that the third roots of 1 in $R(i)$ are: 1, j and $\bar{j}$, where $j = \frac{-1+i\sqrt{3}}{2}$.]

d) Derive explicitly the systems of three variables $S_1(\underline{T}), S_2(\underline{T}), S_3(\underline{T})$ such that for any $\underline{t} := (t_1, t_2, t_2) \in R^3$ the system $S(\underline{t}, X)$ defined in (1) has:

(i) 1 solution in R iff $\underline{t}$ is solution to $S_1(\underline{T})$;

(ii) 2 solutions in R iff $\underline{t}$ is solution to $S_2(\underline{T})$;

(iii) 3 solutions in R iff $\underline{t}$ is solution to $S_3(\underline{T})$.

NB: we do not count multiplicity.

3) Let R be a real closed field and $R(i)$ be its algebraic closure (with $i := \sqrt{-1}$). Consider $P(x) := x^n + p_{n-1}x^{n-1} + \cdots + p_0 \in R[x]$. Show that if all the roots of P in $R(i)$ have non-positive real part, then $Var(p_{n-1}, \ldots, p_0) = 0$.

[Notation: for any $c := r + s\sqrt{-1} \in R(i)$ we say that r is the real part of c and s is the imaginary part of c.]

Please, justify your answers!