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REAL ALGEBRAIC GEOMETRY–WS 2014/15

Exercise Sheet 7

This assignment is due by Tuesday the 9th of December at noon.
Your solutions will be collected during Tuesday's lecture or you can drop them in the postbox 18 near F411.

1) Show that there exist infinitely many non-Archimedean orderings on $\mathbb{Q}(x)$.

2) Prove that for a real closed field R, and $a \in R$ with $a > 0$, the polynomial $p(x) := x^n - a$ has exactly one real root if n is odd and exactly two real roots if n is even.

3) Let R be a real closed field, $n \in \mathbb{N}$ and $\underline{x} := (x_1, \dots, x_n)$. Recall that:

- For any $f \in R[\underline{x}]$, we define $U(f) := \{\underline{x} \in R^n \mid f(\underline{x}) > 0\}$.
- For any $f_1, \dots, f_p \in R[\underline{x}]$ with $p \in \mathbb{N}$, we call *basic open semi-algebraic set* (generated by $f_1, \dots, f_p$) the set
$$U(f_1, \dots, f_p) := \{\underline{x} \in R^n \mid f_1(\underline{x}) > 0, \dots, f_p(\underline{x}) > 0\} = \bigcap_{i=1}^{p} U(f_i)$$

Consider the following topologies:

- *interval topology on* R, i.e. the topology generated by the system of all intervals
$$]a, b[:= \{x \in R \mid a < x < b\}, \quad a, b \in R.$$
- *interval topology on* R^n, i.e. the product topology on R^n given by the interval topology on R.

Prove the following statements:

a) Any polynomial $f \in R[\underline{x}]$ is a continuous mapping from R^n to R w.r.t. the interval topology on R^n.

b) For any $f \in R[\underline{x}]$, $U(f)$ is open in the interval topology on R^n.

c) The basic open semi-algebraic sets form a basis for the interval topology on R^n.

Please, justify your answers!