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REAL ALGEBRAIC GEOMETRY–WS 2014/15

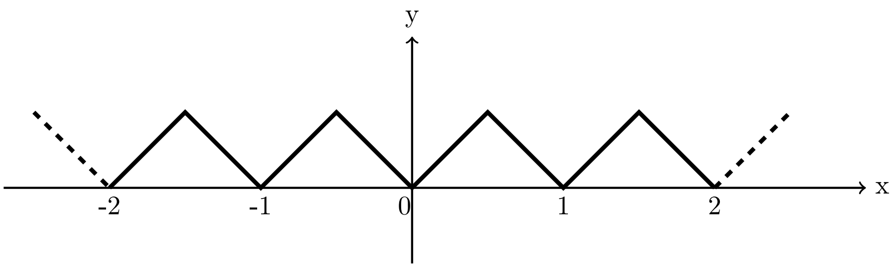
Exercise Sheet 8

This assignment is due by Tuesday the 16th of December at noon. Your solutions will be collected during Tuesday's lecture or you can drop them in the postbox 18 near F411.

1) Let R be a real closed field. Show that the semi-algebraic subsets of R are exactly the finite unions of points and open intervals (bounded or unbounded).

2) Prove the following statements:

a) The infinite zig zag curve $\mathcal{Z}$ in the figure below is not semi-algebraic in $\mathbb{R}^2$.



b) For every compact semi-algebraic subset K of $\mathbb{R}^2$, the intersection of K with the zigzag is semi-algebraic.

3) In the following exercise we will illustrate the Tarski-Seidenberg principle and its proof in a concrete case. Consider the following polynomials in $\mathbb{Z}[T, X]$

$$\begin{aligned} f_1(T,X) &= TX^2 + (T+1)X + 1 \\ f_2(T,X) &= X^3 - 3T^2X + 2T^3. \end{aligned}$$

Prove the following statements:

a) Let $f_2'(T,X)$ be the partial derivative of $f_2(T,X)$ with respect to X. Compute:

- the remainder $g_1(T,X)$ of the euclidean division of $f_2(T,X)$ by $f_1(T,X)$ in $\mathbb{Q}(T)[X]$.
- the remainder $g_2(T,X)$ of the euclidean division of $f_2(T,X)$ by $f_2'(T,X)$ in $\mathbb{Q}(T)[X]$.

b) Compute the X-roots of $f_1(T,X)$, $f_2'(T,X)$, $g_1(T,X)$ and $g_2(T,X)$ in terms of T.

c) As an example, consider a real closed field R and $t \in R$ with $t >> 1$. Denote by $x_1 < x_2 < \ldots < x_5$ the corresponding roots computed in the preceding question, $x_0 := -\infty$, $x_6 := +\infty$, and $I_k :=]x_k, x_{k+1}[$ for $k = 0, \ldots, 5$. Then:

i) compute the sign matrix

$$\mathrm{SIGN}_R(f_1(t,X), f_2'(t,X), g_1(t,X), g_2(t,X)),$$

where each row is given by

$$\Big(\mathrm{sign}(f(I_0)), \mathrm{sign}(f(x_1)), \mathrm{sign}(f(I_1)), \mathrm{sign}(f(x_2)), \ldots, \mathrm{sign}(f(x_5)), \mathrm{sign}(f(I_5))\Big)$$

for f being the corresponding function.

ii) deduce form the part i) the sign matrix

$$\mathrm{SIGN}_R(f_1(t,X), f_2(t,X)).$$

d) Let $\tilde{f}_1(T,X) := T^2 g_1(T,X)$, $\tilde{f}_2(T,X) := g_2(T,X)$, $\tilde{f}_3(T,X) := f_2'(T,X)$ and $\tilde{f}_4(T,X) := f_1(T,X)$. Resume the preceding method (i.e. compute $\tilde{f}_3'$, $\tilde{g}_1, \ldots, \tilde{g}_4$ etc.) until we obtain functions $\tilde{f}_j^{(k)}(T,X)$, $j = 1, \ldots, s_k$ for some step k and number s_k of functions, which do not depend on X anymore.

e) Compute directly the X-roots of $f_2(T,X)$ in terms of T using Cardano formulas and deduce the sign matrix

$$\mathrm{SIGN}_R(f_1(t,X), f_2(t,X)).$$

in terms of t, and verify the preceding results.

<u>NB</u>: the case $t = 0$ has to be treated separately.

f) Conclude that for any real closed field R and any $t \in R$, the resolution of the semi-algebraic system

$$\begin{cases} f_1(t,X) &= tX^2 + (t+1)X + 1 & \rhd_1 & 0 \\ f_2(t,X) &= X^3 - 3t^2X + 2t^3 & \rhd_2 & 0. \end{cases}$$

for some $\rhd_1, \rhd_2 \in \{>, \geq, =, \neq\}$ is equivalent to t solution of a semi-algebraic system only in the variable T.

Please, justify your answers!